Problem 1

1. Compute the limit \( \lim_{x \to \infty} \frac{\ln(x)}{x} \).

2. Compute the limit \( \lim_{x \to \infty} x - \ln(x) \). (Try to find a way to use part 1)

\[
1) \quad \frac{\ln(x)}{x} \quad \xrightarrow{x \to \infty} \quad \frac{\infty}{\infty}, \quad \text{indeterminate}
\]

\[
\text{L'Hopital:} \quad \lim_{x \to \infty} \frac{1}{x} = 0
\]

\[
2) \quad \lim_{x \to \infty} x - \ln(x) = \lim_{x \to \infty} x \left(1 - \frac{\ln(x)}{x}\right)
\]

\[
\lim_{x \to \infty} \left(1 - \frac{\ln(x)}{x}\right) = 1 \quad \text{from part 1}
\]

\[
\Rightarrow \lim_{x \to \infty} x \left(1 - \frac{\ln(x)}{x}\right) = \infty.
\]

\[
\downarrow \quad \downarrow \quad \downarrow 1
\]
Problem 2

Consider the function \( f(x) = x^4 + x^2 - x \).

1. Show that the function has a critical point.
2. Show that the critical point of the function is unique.
3. Starting with \( x_0 = \frac{1}{2} \), apply Newton’s method to generate a better guess \( x_1 \) for the critical point of the function.

1) \( f'(x) = 4x^3 + 2x - 1 \); this is continuous
   \[ f'(-10) = -4000 - 20 - 1 < 0 \]
   \[ f'(10) = 4000 + 20 - 1 > 0 \]
   \( \Rightarrow \) For \( f'(x) = 0 \), a solution exists

2) \( f''(x) = 12x^2 + 2 \); \( \Rightarrow f'(x) \) is differentiable
   By Rolle’s theorem, as \( f'(x) \) is differentiable, if \( f'(c) \)
   is differentiable on \( MVT \)
   has more than one root, \( f''(x) = 0 \) somewhere.
   But \( f''(x) = 12x^2 + 2 \geq 2 \), so \( f''(x) \neq 0 \)
   implies unique critical point.

3) \( x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} \) \( x_0 = \frac{1}{2} \) \( \text{Note:} \) we are solving \( f'(x) = 0 \).
   \( \Rightarrow \) \( x_1 = \frac{1}{2} - \frac{\frac{1}{8} + 1 - 1}{12 \cdot \frac{1}{4} + 2} = \frac{1}{2} - \frac{\frac{12}{8 + 2}}{12 \cdot \frac{1}{4} + 2} = \frac{1}{2} - \frac{1}{5} = \frac{4}{10} = \frac{2}{5} \).