Worksheet # 10 **Name:** Solutions
Date: 03/11/2021
Math 1A: Fall 2021

Problem 1

1. A window is being built and the bottom is a rectangle and the top is a semicircle. If there is 12 metres of material what must the dimensions of the window be to let in the most light?
2. Determine the area of the largest rectangle that can be inscribed in a circle of radius r.

See below.

Problem 2

State whether each of these limit forms are indeterminate or not. If not, evaluate what the limit would be.

I : Indeterminate

1. $\frac{0}{0}$ I
2. $\frac{\infty}{\infty}$ I
3. $\frac{0}{\infty}$ $\to 0$
4. $\frac{\infty}{0}$ $\to \infty$
5. $\infty + \infty$ $\to \infty$
6. $\infty - \infty$ I
7. $\infty \cdot \infty$ $\to \infty$
8. $\infty \cdot 0$ I
9. 0^0 I
10. 0^∞ $\to 0$
11. ∞^0 I
12. 1^∞ I

Problem 3

Evaluate the following limits.

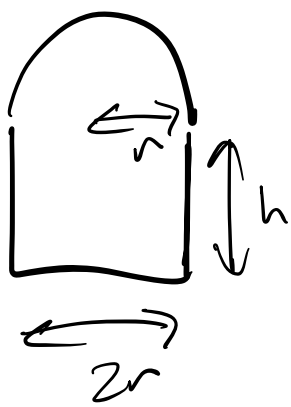
1. $\lim_{x\to 0^+} \frac{1}{x^2} - \cot(x)$
2. $\lim_{x\to 0^+} x^{\sin(x)}$
3. $\lim_{x\to 1} x\ln(x)$
4. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$

Problem [redacted] 4

Find the following limits

1. $\lim_{x\to\infty} x^a - \ln(x)$ for $a > 0$
2. $\lim_{x\to\infty} x^p e^{-x}$ for any p.

1.1)



$\Longrightarrow$ Perimeter: $\pi r + 2r + 2h = 12$

$$\text{Area} = \frac{\pi r^2}{2} + 2r \cdot h$$

$$A(r) = \frac{\pi r^2}{2} + \cancel{2}r \cdot \frac{(12 - \pi r - 2r)}{\cancel{2}}$$

Want to maximize area:

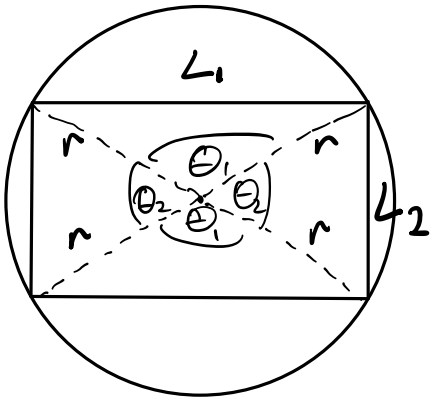
$$\frac{d}{dr} A(r) = \pi r + (12 - \pi r - 2r) + r \cdot (-\pi - 2) = 0$$

$$\Longrightarrow \cancel{\pi r} + 12 - \cancel{\pi r} - 2r - \pi r - 2r = 0$$

$$12 = (4 + \pi) r$$

$$r = \frac{12}{4 + \pi}$$

1.2)



Constraint:

$$2\theta_1 + 2\theta_2 = 2\pi$$

$$\theta_1 + \theta_2 = \pi$$

Area of rectangle is $L_1 \cdot L_2$

Use cosine rule:

$$L_1^2 = r^2 + r^2 - 2r^2\cos(\theta_1)$$
$$= 2r^2(1-\cos(\theta_1))$$
$$L_2^2 = r^2 + r^2 - 2r^2\cos(\theta_2)$$
$$= 2r^2(1-\cos(\pi-\theta_1))$$

Differentiate wrt θ_1:

$$\frac{d}{d\theta_1}L_1 \cdot 2L_1 = 2r^2(\sin(\theta_1))$$

$$\frac{d}{d\theta_1}L_2 \cdot 2L_2 = 2r^2(-\sin(\pi-\theta_1))$$

To maximise: $\frac{d}{d\theta_1}L_1L_2 = L_1\left(\frac{d}{d\theta_1}L_2\right) + \left(\frac{d}{d\theta_1}L_1\right)\cdot L_2 = 0$

$$\Rightarrow \frac{2r^2\sin(\theta_1)\cdot L_2}{2L_1} + \frac{-2r^2\sin(\pi-\theta_1)L_1}{2L_2} = 0$$

$$\Rightarrow \sin(\theta_1) = \sin(\pi - \theta_1)$$

$$\Rightarrow \theta_1 = \frac{\pi}{2}. \quad \text{So } \theta_1 = \theta_2 = \frac{\pi}{2}.$$

So area is maximised when rectangle is in fact a square.

To finish: $L_1^2 = \text{area} = 2r^2\left(1 - \cos\left(\frac{\pi}{2}\right)\right) = \underline{\underline{2r^2}}.$

<u>NOTE:</u> This question is meant to be challenging.

3.1) This $\rightarrow \infty - \infty$, indeterminate.

$$\frac{1}{x^2} - \frac{1}{\tan(x)} = \frac{\tan(x) - x^2}{x^2\tan(x)} \xrightarrow{x \to 0^+} \frac{0}{0}$$

l'hôpital's: $= \lim_{x \to 0^+} \dfrac{\sec^2(x) - 2x}{2x\tan(x) + x^2\sec(x)}$

$$= \infty$$

3.2) This $\rightarrow 0^0$ indeterminate.

Take ln: $\ln y = \sin(x)\ln(x)$

$$\lim_{x \to 0^+} \sin(x)\ln(x) \rightarrow 0 \cdot -\infty, \text{ indeterminate}$$

$$\rightarrow \lim_{x \to 0^+} \frac{\ln(x)}{\operatorname{cosec}(x)} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x}$$

$$= \lim_{x\to 0^+} \frac{-\sin(x)\tan(x)}{x} = \lim_{x\to 0^+} \frac{-\cos(x)\tan(x) - \sin(x)\sec^2(x)}{1}$$

$$= 0.$$

So $\lim_{x\to 0^+} x^{\sin(x)} = e^0 = 1.$

3.3) $\lim_{x\to 1} x\ln(x) = 0.$ This is not indeterminate!

3.4) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \longrightarrow 1^\infty$, indeterminate

take log: $x\ln\left(1+\frac{1}{x}\right)$, $\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \to \infty \cdot 0$, indeterminate

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \cancel{-\frac{1}{x^2}}}{\cancel{-\frac{1}{x^2}}} = 1$$

$$\Longrightarrow \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e^1 = e.$$

4.1) $\lim_{x\to\infty} x^a - \ln(x)$, $a>0$, $\to \infty - \infty$, indeterminate

$$= \lim_{x\to\infty} x^a\left(1 - \frac{\ln(x)}{x^a}\right)$$

$$\lim_{x\to\infty} \frac{\ln(x)}{x^a} = \lim_{x\to\infty} \frac{\frac{1}{x}}{ax^{a-1}} = \frac{1}{a}\cdot\frac{1}{x}\cdot\frac{1}{x^{a-1}}$$

$$= \frac{1}{a}\cdot\frac{1}{x^a} \xrightarrow[x\to\infty]{} 0$$

$$\Rightarrow \lim_{x\to\infty} x^a\left(1-\frac{\ln(x)}{x^a}\right) \to \lim_{x\to\infty} x^a \cdot 1 = \infty.$$

4.2) $\lim_{x\to\infty} x^p e^{-x}$, any p

$$= \lim_{x\to\infty} \frac{x^p}{e^x} \text{ , if } p<0 \text{ , } \longrightarrow \frac{1}{\infty} \to 0$$

if $p>0$, $\to \frac{\infty}{\infty}$, indeterminate

$$= \lim_{x\to\infty} \frac{px^{p-1}}{e^x} \text{ , not helpful.}$$

Different approach: Take log!

$\ln(x^p \cdot e^{-x}) = p\ln(x) - x$, from 1st question $\to -\infty$.

$$\Rightarrow \lim_{x\to\infty} x^p e^{-x} = e^{-\infty} = 0.$$