Quiz # 2 **Name:** Solutions
Date: 25/9/2024
Math 128A - 106: Fall 2024

For full credit, please clearly show all your work.

Problem 1

Consider the fixed point iteration $x_{n+1} = g(x_n)$ *where* $g(x) = 3(\frac{x}{2}+2)^{-1}$ *with initial guess* $x_0 = 0$. *Show that the fixed point iteration converges.*

Need $[a,b]$ where 1) $|g'(x)|<1$
2) $g(x) \in [a,b]$

$x_1 = g(x_0) = \frac{3}{2} \longrightarrow$ Try $[a,b]=[0,2]$

$$g'(x) = 3 \cdot -(\tfrac{x}{2}+2)^{-2} \cdot \tfrac{1}{2} = -\tfrac{3}{2}\frac{1}{(\frac{x}{2}+2)^2} < 0, \text{ if } x > 0$$

$\Rightarrow$ Max/min $g(x)$ at endpoints

$g(0) = \frac{3}{2}$, $g(2) = 3(\frac{2}{2}+2)^{-1} = 1$

$\longrightarrow g(x) \in [0,2]$ if $x \in [0,2]$

$g''(x) = \frac{3}{2}\frac{1}{(\frac{x}{2}+2)^3}$, increasing for $x \geq 0$

$\Rightarrow$ Max/min $g'(x)$ at endpoints

$g'(0) = -\frac{3}{2}\cdot\frac{1}{2^2} = -\frac{3}{8}$ $\qquad$ $g'(2) = -\frac{3}{2}\cdot\frac{1}{3^2} = \frac{-1}{6}$

$\longrightarrow |g'(x)|<1$ if $x \in [0,2]$

$\Longrightarrow$ Fixed point converges w/ $x_0 = 0$

Problem 2

Consider the sequence $\{x_n\}$ *where* $x_n = \frac{1+n}{2n^3}$.

1. *Find the limit as* $n \to \infty$.
2. *Find the order of convergence. (Recall: If* $\lim_{n\to\infty} \frac{|x_{n+1}-x|}{|x_n-x|^\alpha} = C$*, what is the order of convergence?)*

1) $\lim_{n\to\infty} x_n = 0$

2) $$\lim_{n\to\infty} \frac{2+n}{2(n+1)^3} \cdot \left(\frac{2n^3}{1+n}\right)^\alpha$$

$$= \lim_{n\to\infty} \frac{2^\alpha (2+n)\, n^{3\alpha}}{2(n+1)^{3+\alpha}}$$, only care about highest powers

$$= \lim_{n\to\infty} \frac{2^\alpha n^{3\alpha+1}}{2n^{3+\alpha}}$$

$$= \begin{cases} 0, & \alpha < 1 \\ 1, & \alpha = 1 \\ \infty, & \alpha > 1 \end{cases}$$ $\Rightarrow$ Linearly convergent