ODEs (Ordinary Differential Equations)

$$\dot{y} = \frac{dy}{dt} = f(y(t), t)$$

→ Governs processes in time

From Calculus:

$$\dot{y} = ky \quad , \quad k \text{ constant}$$

$$\Rightarrow y(t) = y(0) e^{kt}$$

BUT what if more complicated?

$$\dot{y} = \sin(y) \quad , \quad \dot{y} = \underbrace{NN}_{\text{Neural Network}}(y)$$

→ Solve numerically.

Q) When does solution exist?

Def. Lipschitz continuity

$f(x)$ is Lipschitz continuous on a domain $[a,b]$ if $\forall x_1, x_2 \in [a,b]$

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$$

for some constant L.

Thm. Picard-Lindelöf

If f is Lipschitz, $\dot{y} = f(y)$ has a unique solution.

eg: $f(x)$, where $f'(x)$ exists on $[a,b]$,

let $L = \max_{x \in [a,b]} f'(x)$,

then $|f(x_1) - f(x_2)| \leq L |x_1 - x_2|$ by MVT.

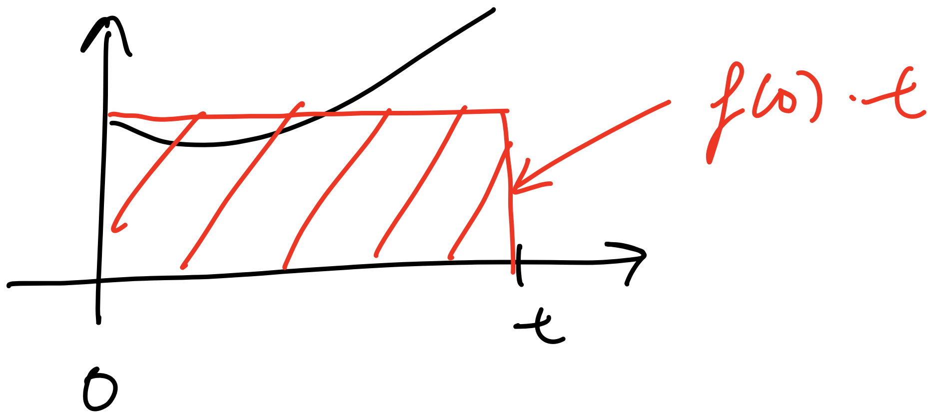
How to solve ODEs numerically?

FTC: $\dot{y} = f(y)$

$$\Rightarrow y(t) = y(0) + \underbrace{\int_0^t f(s)\, ds}_{\text{integrate using numerical integration}}$$

Most simple method:

$$\int_0^t f(s)\,ds \approx f(0)\cdot t$$



<u>i.e.</u> quadrature rule for $\int_0^1 f(x)\,dx$

$$\{x_i\} = \{0\}, \quad \{w_i\} = \{1\}$$

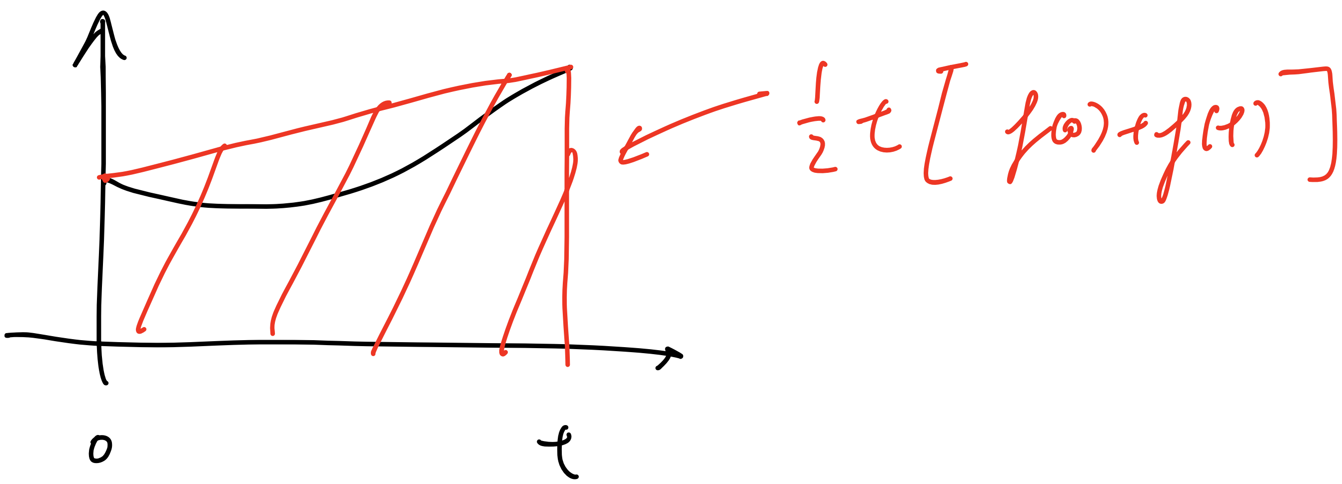
$$\Longrightarrow \quad y(t) = y(0) + t\,y(0)$$

$$= y(0)\,(1+t)$$

$\longrightarrow$ Forward Euler method

Another method:

$$\int_0^t f(s)\,ds \approx t\left[\frac{1}{2}f(0)+\frac{1}{2}f(t)\right]$$



$$\Longrightarrow\quad y(t) = y(0) + \frac{t}{2}\left[y(0)+y(t)\right]$$

$$y(t) = \frac{y_0\left(1+\frac{t}{2}\right)}{1-\frac{t}{2}}$$

$\longrightarrow$ Trapezoidal rule,

(AKA Crank-Nicholson)