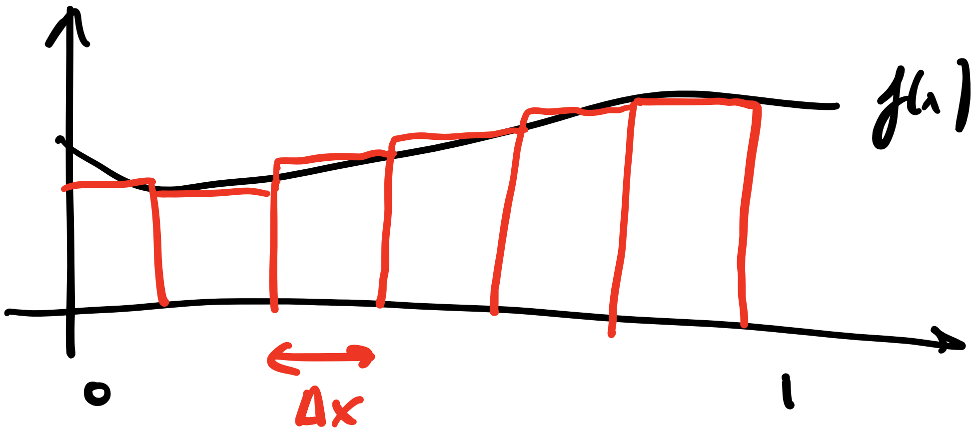
Numerical Integration

Want to calculate $\int_0^1 f(x)\,dx = F$

Riemann sum:



$$F \approx \sum f(x_i)\,\Delta x \quad , \; x_i = i\,\Delta x$$

→ Can we do better?

Quadrature

$$F = \sum_{k=1}^{n} f(x_k)\, w_k$$

quadrature weights

quadrature pts

Def. Precision of quadrature rule

For a given set of $\{x_k\}$, $\{w_k\}$,

what is the highest degree p of polynomial

s.t. $$\int_0^1 p(x)\,dx = \sum_{k=1}^{n} p(x_k)\, w_k$$

is exact?

GOAL: Find $\{x_k\}$, $\{w_k\}$ s.t. precision as high as possible

Example Find quadrature rule of highest precision possible using $n=1$.

Deg 0: $\int_0^1 1 \, dx = 1$

$$\sum_{k=1}^{1} p(x_k) w_k = w_k = 1$$

Deg 1: $\int_0^1 x \, dx = \frac{1}{2}$

$$\sum_{k=1}^{1} p(x_k) w_k = x_1 \cdot w_1 = x_1 = \frac{1}{2}$$

Deg 2: $\int_0^1 x^2 \, dx = \frac{1}{3}$

$$\sum_{k=1}^{1} p(x_k) w_k = x_1^2 \cdot w_1 = \frac{1}{4} \neq \frac{1}{3}$$

$\longrightarrow$ So $\{x_1\} = \{\frac{1}{2}\}$, $\{w_1\} = \{1\}$

is quadrature rule of highest precision,

precision $= 1$

Q) General, for n points and weights, what is highest possible precision?

A) $2n-1$. Why?

Deg. $2n-1$ polynomial:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n-1} x^{2n-1}$$

$2n$ a_k degrees of freedom

n points $+ n$ weights $= 2n$ degrees of freedom

$\rightarrow$ $2n-1$ max. possible freedom w/ n points and n weights

(*) This has a special name:

Gauss - Legendre Quadrature

$\rightarrow$ n GL points can integrate all polynomials of deg. up to $2n-1$

$\rightarrow$ This is really important in practice, it's a huge part of how real computations work