Numerical Differentiation

- Given $f(x)$, how to approximate $f'(x)$?
  - e.g. For Newton, Gradient Descent, etc...

2 Equivalent ways to see it:

(Give same answer)

1) Given $(x_1, f(x_1)), \ldots, (x_n, f(x_n))$

Use polynomial interpolation to get $p(x) \approx f(x)$,

$\longrightarrow$ Differentiate to get $p'(x) \approx f'(x)$

2) Taylor series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \ldots$$

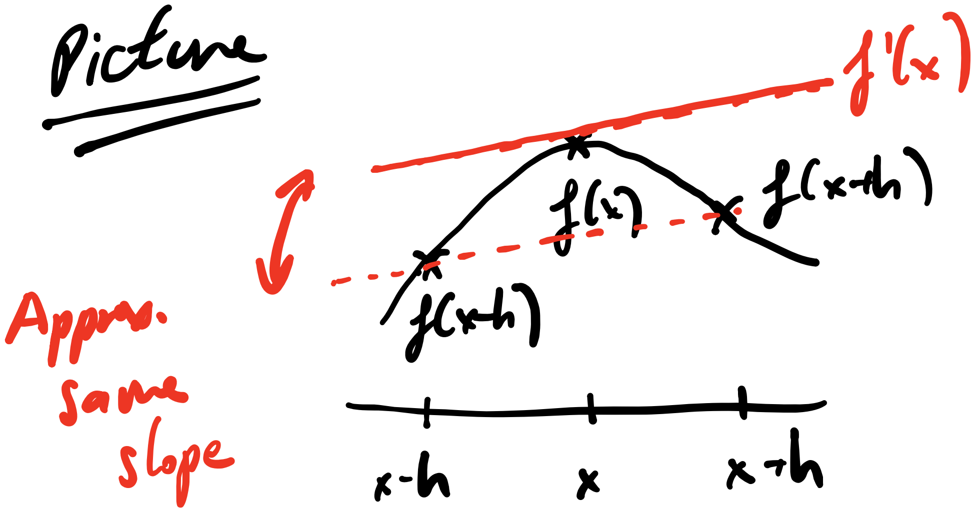
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \ldots$$

$$\Rightarrow f(x+h) - f(x-h) = 2h f'(x) + \frac{h^3}{3} f'''(x) + \dots$$

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{h^2}{6} f'''(x) + \dots$$

As $h \to 0$, converges to $f'(x)$, with order $O(h^2)$.

<u>Picture</u>



Ex. Find approx. to $f''(x)$ using $f(x)$, $f(x+h)$, $f(x-h)$. What the order of accuracy?

Have:

① $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{iv}(x) + \dots$

② $f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{iv}(x) + \dots$

③ $f(x)$

$a①+b②+c③$

$$= (a+b+c)f(x) + (a-b)hf'(x) + (a+b)\frac{h^2}{2}f''(x) + (a-b)\frac{h^3}{6}f'''(x) + (a+b)\frac{h^4}{24}f^{iv}(x) + \dots$$

Want: $(a+b+c) = 0$

$(a-b) = 0 \quad \Rightarrow \quad a = b = 1$

$(a+b) = 2 \qquad\qquad c = -2$

$$\Rightarrow f(x-h) - 2f(x) + f(x+h) = h^2 f''(x) + \frac{h^4}{12} f^{iv}(x) \dots$$

So
$$\frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x) + O(h^2)$$

Say: $f(x) = e^x$, approximate w/

1) $x = 0, \; h = 1$

$$\Rightarrow \frac{f(-1) - 2f(0) + f(1)}{1^2} = e^{-1} - 2 + e \approx 1.08616\dots$$

4x smaller

2) $x = 0, \; h = \frac{1}{2}$

$$\frac{f(-\frac{1}{2}) - f(0) + f(\frac{1}{2})}{(1/2)^2} = 4\left(e^{-\frac{1}{2}} - 2 + e^{\frac{1}{2}}\right) \approx 1.0210\dots$$