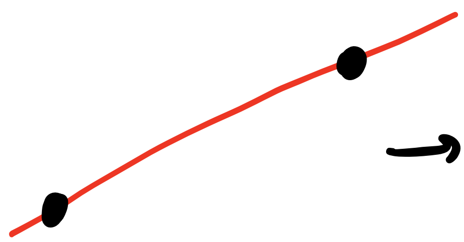
Polynomial Interpolation

- How people approximate things (smoothly) in real life

Fact about polynomials

Given n points, ~~unique~~ degree $n-1$ polynomial is defined

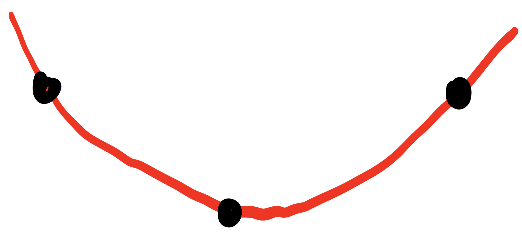
eg: 2 pts



straight line

$\rightarrow a_0 + a_1 x$

3 pts



quadratic function

$\rightarrow a_0 + a_1 x + a_2 x^2$

- each point is one degree of freedom

Lagrange interpolation

Easiest way to define

Q) Given $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ find degree $n-1$ polynomial going through every point

A) Define $L_i(x)$, where

Kronecker delta

$$L_i(x_j) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} = \delta_{ij}$$

Then $p(x) = \sum_{i=1}^{n} y_i L_i(x)$,

because $p(x_j) = \sum_{i=1}^{n} y_i \underbrace{L_i(x_j)}_{\neq 0 \text{ only if } i=j}$

$\Rightarrow p(x_j) = y_j$, as we want.

How to generate $L_i(x)$?

Lagrange Interpolant

$$L_i(x) = \prod_{k \neq i} \frac{(x - x_k)}{(x_i - x_k)}$$

e.g. for 2 points (x_1, y_1), (x_2, y_2)

$$L_1(x) = \frac{x - x_2}{x_1 - x_2},$$

then if $x = x_1$, $L_1(x_1) = \frac{x_1 - x_2}{x_1 - x_2}$, cancels

$$L_1(x_2) = \frac{x_2 - x_2}{x_1 - x_2}$$

$= 0$ on top

$\neq 0$ on bottom

Example 1) Find quadratic interpolant to $\sin(x)$ using points at $x = \{0, \frac{\pi}{2}, \pi\}$

2) Use to approximate $\sin\left(\frac{\pi}{3}\right)$

---

1) $\sin(0) = 0 \rightarrow (x_1, y_1) = (0, 0)$

$\sin\left(\frac{\pi}{2}\right) = 1 \rightarrow (x_2, y_2) = \left(\frac{\pi}{2}, 1\right)$

$\sin(\pi) = 0 \rightarrow (x_3, y_3) = (\pi, 0)$

$$L_2(x) = \frac{(x - x_1)}{(x_2 - x_1)} \cdot \frac{(x - x_3)}{(x_2 - x_3)}$$

$$= \frac{x}{\pi/2} \cdot \frac{x - \pi}{\frac{\pi}{2} - \pi}$$

$$= \frac{x}{\pi/2} \cdot - \frac{x-\pi}{\pi/2}$$

$$= \frac{4}{\pi^2} x (\pi - x)$$

$$\Rightarrow p_2(x) = \frac{4}{\pi^2} x (\pi - x)$$

Quadratic approximation

2) $\sin(\frac{\pi}{3}) \approx p_2(\frac{\pi}{3})$

$$p_2(\frac{\pi}{2}) = \frac{4}{\pi^2} \cdot \frac{\pi}{3} (\pi - \frac{\pi}{3})$$

$$= \frac{4}{3} \cdot \frac{1}{\pi} \left(\frac{2\pi}{3} \right)$$

$$= \frac{8}{9} = 0.888....$$

$\sin(\frac{\pi}{3}) = 0.866...$, so fairly close

Divided differences (DD)

- Another way to do polynomial interpolation
  - You'll see an application next time

- Won't explain right now, just tell you how to do it

- DD table

x	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$
x_0	$f[x_0]$		
		$f[x_0, x_1]$	
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$
		$f(x_1, x_2)$	
x_2	$f[x_2]$		$\vdots$
		$\vdots$	
x_3	$f[x_3]$		
	$\vdots$		

- top are coefficients

Where:

$$f[x_1, \dots, x_n] = \frac{f[x_2, \dots, x_n] - f[x_1, \dots, x_{n-1}]}{x_n - x_1}$$

$$f[x_i] = f(x_i)$$

Example $(x_i, y_i) = \left\{ (0,0), \left(\frac{\pi}{2}, 1\right), (\pi, 0) \right\}$

Same as before.

x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$
0	0		
		$\frac{1-0}{\frac{\pi}{2}-0} = \frac{1}{\pi/2}$	
$\frac{\pi}{2}$	1		$\frac{\frac{-1}{\pi/2} - \frac{1}{\pi/2}}{\pi - 0} = \frac{-4}{\pi^2}$
		$\frac{0-1}{\pi - \frac{\pi}{2}} = \frac{-1}{\pi/2}$	
π	0		

$$\Rightarrow p_2(x) = 0 + \frac{1}{\pi/2}x - \frac{4}{\pi^2}x\left(x - \frac{\pi}{2}\right)$$

$$= \frac{2}{\pi}x - \frac{4}{\pi^2}x^2 + \frac{2}{\pi}x$$

$$= \frac{4}{\pi}x - \frac{4}{\pi^2}x^2$$

$$= \frac{4}{\pi^2}x\left(\pi - x\right)$$

$\rightarrow$ Exactly the same as before.