Order of convergence - Fixed Point Iteration

Recall: $x_{n+1} = g(x_n)$, at limit $x = g(x)$,

Order of convergence for sequences:

$\{x_n\} \longrightarrow x$, $\lim_{n\to\infty} \frac{|x_{n+1}-x|}{|x_n-x|^\alpha} = A$

$A > 0$, then $\{x_n\} \to x$

with order α.

Example $p_n = 2^{-n}$, $\lim_{n\to\infty} p_n = 0$

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^{\alpha}} = \lim_{n\to\infty} \frac{2^{-n-1}}{2^{-\alpha n}}$$

$$= \lim_{n\to\infty} 2^{\alpha n - n - 1}$$

$$= \begin{cases} 0, & \alpha < 1 \\ \frac{1}{2}, & \alpha = 1 \\ \infty, & \alpha > 1 \end{cases}$$

$\longrightarrow$ Linearly convergent.

How to maybe accelerate fixed point without doing Newton?

1) Aitken

2) Steffensen

NOTE: Nobody actually really does this....

Aitken: If $\{x_n\} \to x$ linearly,

$$\lim_{n\to\infty} \underbrace{\frac{x_{n+1}-x}{x_n-x}}_{c_n} = c$$

Basically compute $c_{n+1} - c_n$, as

$$\lim_{n\to\infty} c_{n+1} - c_n = 0$$

$$\Rightarrow \quad 0 \cong \frac{x_{n+2} - x}{x_{n+1} - x} - \frac{x_{n+1} - x}{x_n - x}, \text{ solve for } x$$

$$\Rightarrow \quad \hat{x}_n = A\left(x_n, x_{n+1}, x_{n+2}\right) = \frac{x_n x_{n+2} - x_{n+1}^2}{x_n - 2x_{n+1} + x_{n+2}}$$

So given $\{x_0, x_1, x_2, x_3, \ldots\}$ from fixed point,

Compute $\hat{x}_0$ $\hat{x}_1$ .... and so on

Steffensen: How to improve Aitken?

IDEA One step of Aitken only needs two of p_0, p_1 to work

→ 1) Given p_0

2) Compute $p_1 = g(p_0)$

3) Aitken: $\hat{p}_0 = \text{Aitken}(x_0, x_1, x_2)$

4) Set $p_0 = \hat{p}_0$, and go to step (1)