Root Finding

Goal: Given $f(x)$, want to find some x s.t. $f(x)=0$.

→ This x is called a root.

- Bisection method

- Fixed point iteration

Bisection Algorithm

0) Given $f(x)$ continuous on $[a,b]$, $f(a)\cdot f(b)<0$

1) Compute $f\left(\frac{a+b}{2}\right)$

midpoint

2) If $f\left(\frac{a+b}{2}\right)\cdot f(a) > 0$,

same sign

set $a \to \frac{a+b}{2}$,

(ie. shrink domain to $\left[\frac{a+b}{2}, b\right]$)

else

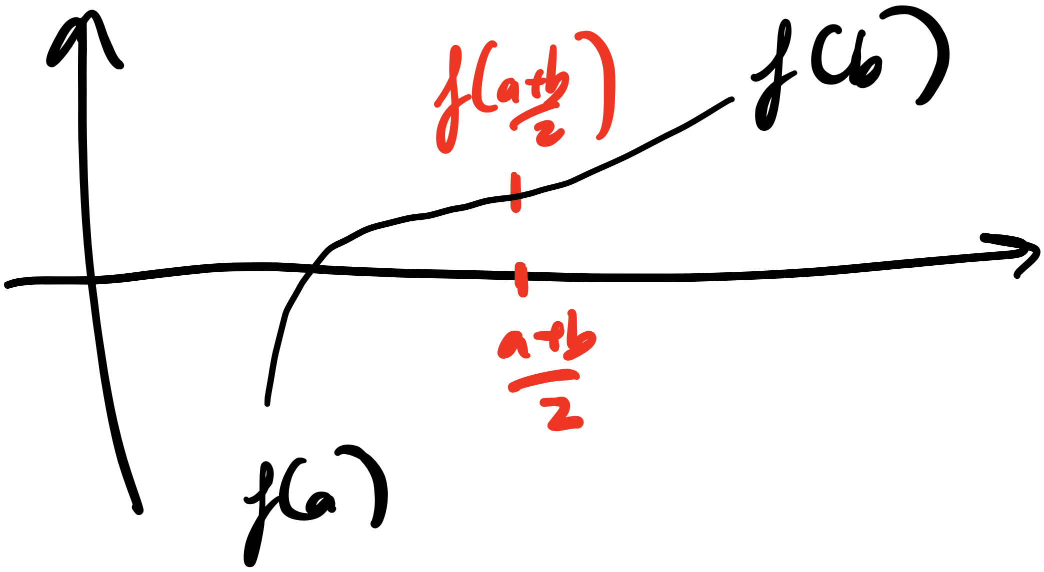
set $b \to \frac{a+b}{2}$,

(ie. shrink domain to $\left[a, \frac{a+b}{2}\right]$

3) Go back to step 1

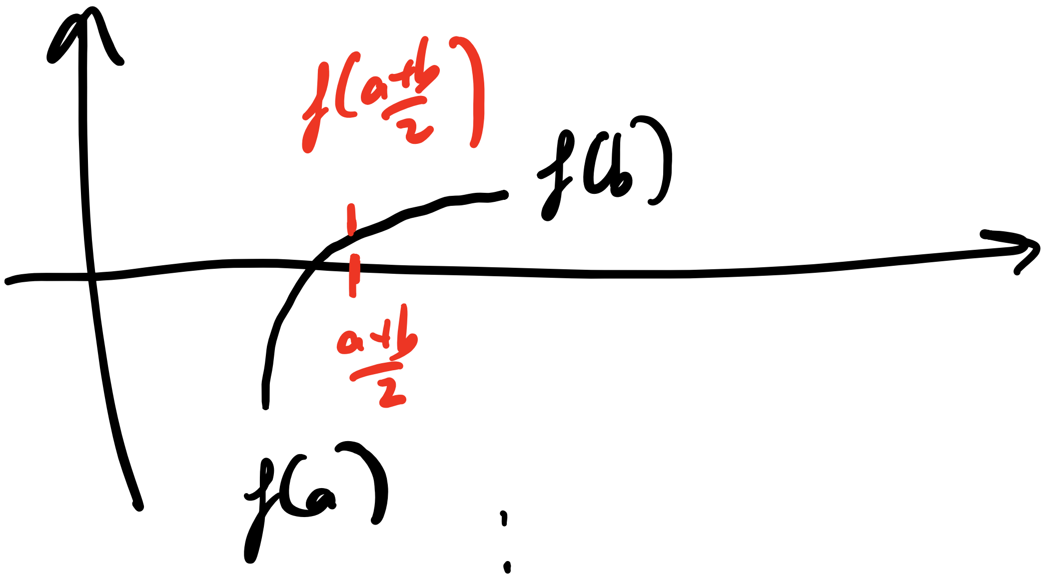
- Repeat until $|b-a| <$ tolerance

Bisection picture



Iteration 1

$\downarrow$ set $b \to \frac{a+b}{2}$



Iteration 2

⋮

until gap between $[a,b]$ small enough

Fixed Point Iteration (Give examples next time)

Def. Fixed Point

Given some function g, x is a fixed pt if

$$g(x) = x.$$

Def. Fixed point iteration,

$$x_{n+1} = g(x_n),$$

hope that as you iterate eventually reach $x = g(x)$.

IDEA: To solve $f(x) = 0$, find some $g(x)$

s.t. $$f(x_0) = 0 \iff g(x_0) = x_0$$

e.g. $f(x) = 0$

$f(x) + x = x$, call $g(x) = f(x) + x$

So when does $x_{n+1} = g(x_n)$ converge?

Theorem Banach fixed point theorem

If g differentiable on $x \in [a,b]$, and

1) $g(x) \in [a,b]$ for $x \in [a,b]$ (maps to self)

2) $|g'(x)| \leq c < 1$ for $x \in [a,b]$ (contraction)

then $x_{n+1} = g(x_n)$ converges.

<u>Why?</u> $x_{n+1} = g(x_n)$

at limit : $x = g(x)$

(subtract)

$$\Rightarrow \underbrace{x - x_{n+1}}_{e_{n+1}} = g(x) - g(x_n)$$

e_{n+1} ← error at nel step

$$\rightarrow |e_{n+1}| = |g(x) - g(x_n)|$$

Apply MVT

$$|e_{n+1}| = \underbrace{|g'(\xi)|}_{\leq c < 1,\ \text{contraction}}\ \underbrace{|x - x_n|}_{e_n}$$

$$\Rightarrow |e_{n+1}| \leq c|e_n|$$

$$\Rightarrow |e_n| \leq c^n |e_0|,$$

as $n \to \infty$, as $c < 1$, $c^n \to 0$,

$$\Rightarrow \lim_{n\to\infty} |e_n| \to 0.$$

OR

$$\lim_{n\to\infty} |x - x_n| \to 0$$

so $x = x_n$.

e.g. $x_{n+1} = \frac{1}{2}\left(\frac{2}{x_n} + x_n\right)$

a) Find limit b) Show convergence

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a) At limit, $x = g(x)$

$\Rightarrow x = \frac{1}{2}\left(\frac{2}{x} + x\right)$, $x = \sqrt{2}$

b) Need to show 2 conditions

$$g'(x) = \frac{1}{2}\left(-\frac{2}{x^2} + 1\right)$$

$$g''(x) = \frac{1}{2}\left(\frac{4}{x^3}\right)$$

$\lim_{x \to \infty} g'(x) = \frac{1}{2}$, $g'(1) = -\frac{1}{2}$

Also, $g''(x) > 0$ if $x > 0$,

$\Rightarrow$ On interval $[1, \infty)$, $|g'(x)| < 1$

(Contraction satisfied)

Check self map:

$\rightarrow$ check $[\min g, \max g] \subseteq [1, \infty)$

Critical Pts: $g'(x) = 0$, $\rightarrow x = \sqrt{2}$

Endpoints: $x = 0, \infty$

Check: $g(1) = \frac{3}{2}$

$g(\sqrt{2}) = \sqrt{2}$

$g(\infty) = \infty$

So self map satisfied

$\rightarrow$ Converges by theorem

Newton's method

Q) What's a good choice of $g(x)$ in general to solve $f(x)=0$?

A) Newton says:

to solve $f(x)=0$,

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$