Numerical Linear Algebra

Basically, solving $\underline{\underline{A}}\,\underline{x} = \underline{b}$ on the computer.

Matrix Vector

Please review:

1) Row reduction
2) Inner products
3) Gram-Schmidt
4) Eigenvalues

Why? It's how things are actually calculated in real life.

So far in this class:

- Fixed pt : $x_{n+1} = f(x_n)$

- Polynomial interpolation : $y(x) = \sum L_i(x) y_i$

- Numerical Diff./Int. $f'(x_0) \,/\, \int_0^1 f(x)\,dx$

$\Rightarrow$ All functions of one variable.

In real life, functions have many variables.

eg: Newton's method

1D $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$n > 1$

nD $\underline{x}_{n+1} = \underline{x}_n - J(x_n)^{-1} f(\underline{x}_n)$

vector

Matrix inverse

- Jacobian

eg. $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sin(x_1+x_2) \\ \cos(x_2) \end{pmatrix}$

$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \cos(x_1+x_2) & \cos(x_1+x_2) \\ 0 & -\sin(x_2) \end{pmatrix}$

- In practice $n \sim 10^3 - 10^6$,
  you can't compute this analytically
  or by hand

What do we care about?

- Cost ← Focus
- Stability ← Won't really cover in this class

Basic operations:

1) Dot product

$$\underline{x} \cdot \underline{y} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

n multiplications, $n-1$ additions

- $O(n)$ cost

2) Matrix - vector multiplication

$$\underline{\underline{A}}\,\underline{x} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ \vdots & & \ddots \end{pmatrix}}_{m \times n} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{n \times 1} = \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}}_{m \times 1}$$

$\Rightarrow$ m dot products of size n

$\Rightarrow O(mn)$

3) Matrix - matrix multiplication

$$\underline{\underline{A}}\,\underline{\underline{B}} = \underbrace{\begin{pmatrix} a_{11} & \cdots \\ \vdots & \ddots \end{pmatrix}}_{m \times n} \underbrace{\begin{pmatrix} b_{11} & \cdots \\ \vdots & \ddots \end{pmatrix}}_{n \times p}$$

$O(mnp)$. Can you see why?

4) Solution. $\underline{\underline{A}}\,\underline{x} = \underline{b}$, given $\underline{\underline{A}}$, $\underline{b}$, want to find $\underline{x}$.

Idea: 1) Matrix Inverse X NO

2) Gauss-Elimination ✓ YES

- Row reduction

e.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

1 $\downarrow$ row 2 -= 3 x row1

$$\underbrace{\begin{pmatrix}1 & 0\\ -3 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}}_{\begin{pmatrix}1 & 2\\ 0 & -2\end{pmatrix}}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 0\\ -3 & 1\end{pmatrix}\begin{pmatrix}2\\ 3\end{pmatrix}}_{=\begin{pmatrix}2\\ -3\end{pmatrix}}$$

$$\Longrightarrow \begin{pmatrix}1 & 2\\ 0 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}2\\ -3\end{pmatrix}$$

$\downarrow$ 2 Now can solve

– back-substitution

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ -2x_2 &= -3 \end{aligned} \quad\Longrightarrow\quad \begin{aligned} x_1 &= -1 \\ x_2 &= \tfrac{3}{2} \end{aligned}$$

Overall:

Step 1:

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}}_{\text{lower triangular}} A = \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}}_{\text{upper triangular}}$$

$\Rightarrow$ $\underline{\underline{A}} = \underline{\underline{L}}\,\underline{\underline{U}}$ factorise

- Called LU factorisation

Step 2: Back substitution

$$\underline{\underline{U}}\,\underline{x} = \underline{\underline{L}}\,\underline{b}$$

$\rightarrow$ solve x_n, then solve $x_{n-1}, \dots, x_1$ in order

What is the cost?

(1) When you row reduce:

e.g. $\begin{pmatrix} 1 & \cdots \\ 2 & \cdots \\ 3 & \cdots \\ 4 & \cdots \end{pmatrix}$

you do:

row 2 − 2 × row 1
row 3 − 3 × row 1
row 4 − 4 × row 1

so $n-1$ rows where you subtract a row

$\Rightarrow O(n^2)$ for this part,

→ BUT, have to do this for every row,

$\Rightarrow O(n^3)$ cost overall

(2) Backsubstitution:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$$
$$a_{nn}x_n = b_n$$

To solve: $a_{nn}x_n = b_n$

$$\longrightarrow x_n = b_n / a_{nn},$$

1 divide.

$$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$$

$$\Longrightarrow x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

1 multiply, 1 division, 1 subtraction

$$\vdots$$

$$a_{kk} x_k + \cdots + a_{kn} x_n = b_k$$

$$\Rightarrow x_k = \frac{b_k - a_{k,k+1} x_{k+1} - \cdots - a_{kn} x_n}{a_{kk}}$$

$n-k$ multiply, $n-k$ adds,

1 division $\Rightarrow O(n-k)$ operations

$\Rightarrow$ Overall cost for back-substitution

$O(n^2)$