

More ODEs

$$y' = f(t, y(t))$$

$$\longrightarrow y(t) = y(0) + \int_0^t f(s, y(s)) ds$$

Generally, pick step size h ,

Start with $y(0)$, get $y(h)$

$$y(h) \rightarrow y(2h) \rightarrow y(3h)$$

and so on.

Runge-Kutta methods (RK)

Want $y(t) \rightarrow y(t+h)$

IDEA: Get $k_1 = f\left(t + c_1 h, y(t) + d_{11} k_1 + d_{12} k_2 + \dots + d_{1n} k_n\right)$
:
:

$$k_n = f\left(t + c_n h, y(t) + d_{n1} k_1 + \dots + d_{nn} k_n\right)$$

and $y(t+h) = y(t) +$

$$h \left[\beta_1 k_1 + \dots + \beta_n k_n \right]$$

i.e. this approximates $\int_t^{t+h} f(s, y(s)) ds$

→ If you think about it, it's a sort of
a quadrature rule w/

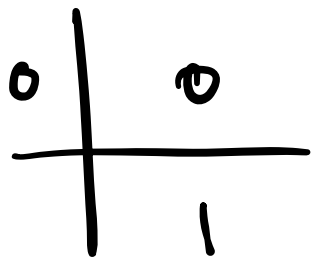
$$pts = \{ t + c_1 h, \dots, t + c_n h \}$$

$$wts = \{ \beta_1, \dots, \beta_n \}$$

People write it in table form:

c_1	d_{11}	d_{12}	\dots	d_{1n}
c_2	d_{21}	d_{22}	\dots	d_{2n}
\vdots				
\vdots				
c_n	d_{n1}	d_{n2}	\dots	d_{nn}
	β_1	β_2	\dots	β_n

Exp.



$$k_1 = f(t, y(t))$$

$$y(t+h) = y(t) + hf(t, y(t))$$

⇒ Euler's method

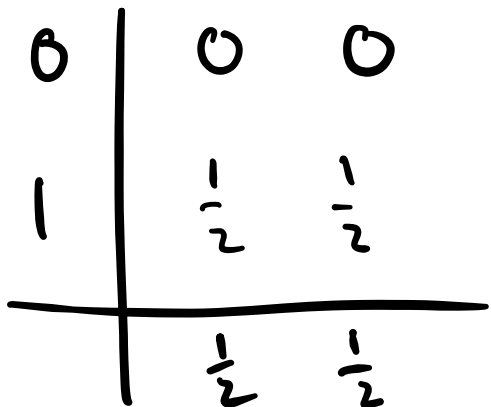


$$k_1 = f(t+h, y+hk_1)$$

$$y(t+h) = y(t) + hk_1$$

⇒ **Implicit** Euler's method

⏟
B/C k_1 depends on k_1



$$k_1 = f(t)$$

$$k_2 = f\left(t+h, y_n + \frac{k_1}{2} + \frac{k_2}{2}\right)$$

$$y(t+h) = y(t) + h\frac{k_1}{2} + h\frac{k_2}{2}$$

⇒ Trapezoidal Rule (Implicit)

Implicit vs. Explicit

0	0	0	0
1	1	0	0
2	1	1	0
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	.	.	.

$$\Rightarrow k_1 = f(t, y(t))$$

$$k_2 = f(t+h, y(t)+hk_1)$$

$$k_3 = f(t+2h, y(t)+hk_1 + hk_2)$$

No k_i depends on itself

\Rightarrow explicit, can compute

k_i sequentially

1	1	0
1	0	1
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	.	.

$$\Rightarrow k_1 = f(t+h, y_n + hk_1)$$

$$k_2 = f(t+h, y_n + hk_2)$$

k_i depend on themselves

\Rightarrow implicit

Explicit if $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & & & \\ & & & \\ & & & \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix}$

matrix is strict lower diagonal

Otherwise RK method is implicit.

How to solve implicit RK?

eg. $\frac{1}{1}$

$$k_1 = f(t+h, y(t)+hk_1)$$
$$y(t+h) = y(t) + hk_1$$

Answer: Fixed point iteration

$$k_1 = f(t+h, y(t) + hk_1)$$

eg. $y'(t) = \sin(y)$, $f(t, y(t)) = \sin(y(t))$

$$k_1 = f(t+h, y(t) + hk_1)$$

$$\Rightarrow k_1 = \sin(y(t) + hk_1)$$

If doing Newton: $k_1 - \sin(y(t) + hk_1) = 0$

$$\Rightarrow k_1^{n+1} = k_1^n - \frac{k_1^n - \sin(y(t) + hk_1^n)}{1 + h \cos(y(t) + hk_1^n)}$$