More ODEs

$$y' = f(t, y(t))$$

$$\longrightarrow \quad y(t) = y(0) + \int_0^t f(s, y(s))\, ds$$

Generally, pick step size h,

Start with $y(0)$, get $y(h)$

$$y(h) \to y(2h) \to y(3h)$$

and so on.

Runge-Kutta methods (RK)

Want $y(t) \to y(t+h)$

IDEA: Get
$$k_1 = f\left(t + c_1 h,\; y(t) + \alpha_{11} k_1 + \alpha_{12} k_2 + \dots + \alpha_{1n} k_n\right)$$

$$\vdots$$

$$k_n = f\left(t + c_1 h,\; y(t) + \alpha_{n1} k_1 + \dots + \alpha_{nn} k_n\right)$$

and
$$f(t+h) = f(t) + \underbrace{h\left[\beta_1 k_1 + \dots + \beta_n k_n\right]}$$

i.e. this approximates $\int_t^{t+h} f(s, y(s))\, ds$

→ If you think about it, it's a sort of a quadrature rule w/

$$\text{pts} = \{ t + c_1 h, \dots, t + c_n h \}$$

$$\text{wts} = \{ \beta_1, \dots, \beta_n \}$$

People write it in table form:

$$\begin{array}{c|cccc} c_1 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ c_2 & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & & \\ c_n & \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \\ \hline & \beta_1 & \beta_2 & \cdots & \beta_n \end{array}$$

Ex.

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

$$k_1 = f(t, y(t))$$

$$y(t+h) = y(t) + h f(t, y(t))$$

$\Rightarrow$ Euler's method

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

$$k_1 = f(t+h, y+hk_1)$$

$$y(t+h) = y(t) + hk_1$$

$\Rightarrow$ $\underbrace{\text{Implicit}}_{\text{B/C } k_1 \text{ depends on } k_1}$ Euler's method

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$k_1 = f(t)$$

$$k_2 = f\left(t+h, y_n + \frac{k_1}{2} + \frac{k_2}{2}\right)$$

$$y(t+h) = y(t) + \frac{hk_1}{2} + \frac{hk_2}{2}$$

$\Rightarrow$ Trapezoidal Rule (Implicit)

Implicit vs. Explicit

$$
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
2 & 1 & 1 & 0 \\
\hline
 & \cdot & \cdot & \cdot
\end{array}
\quad \Rightarrow \quad
\begin{aligned}
k_1 &= f(t, y(t)) \\
k_2 &= f(t+h, y(t)+hk_1) \\
k_3 &= f(t+2h, y(t)+hk_1 + hk_2)
\end{aligned}
$$

No k_i depends on itself

$\Rightarrow$ explicit, can compute k_i sequentially

$$
\begin{array}{c|cc}
1 & 1 & 0 \\
1 & 0 & 1 \\
\hline
 & \cdot & \cdot
\end{array}
\quad \Rightarrow \quad
\begin{aligned}
k_1 &= f(t+h, y_n+hk_1) \\
k_2 &= f(t+h, y_n+hk_2)
\end{aligned}
$$

k_i depend on themselves

$\Rightarrow$ implicit

Explicit if $A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ & & \vdots & \\ \alpha_{m1} & \cdots & \cdots & \alpha_{mm} \end{pmatrix}$

matrix is strict lower diagonal

Otherwise RK method is implicit.

<u>How to solve implicit RK?</u>

<u>eg.</u>

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

$$k_1 = f(t+h, y(t)+hk_1)$$

$$y(t+h) = y(t) + hk_1$$

Answer: Fixed point iteration

$$k_1 = f(t+h,\ y(t) + h k_1)$$

e.g. $y'(t) = \sin(y)$, $f(t, y(t)) = \sin(y(t))$

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \quad , \quad k_1 = f(t+h,\ y(t) + h k_1)$$

$$\Rightarrow k_1 = \sin(y(t) + h k_1)$$

If doing Newton: $k_1 - \sin(y(t) + h k_1) = 0$

$$\Rightarrow k_1^{n+1} = k_1^n - \frac{k_1^n - \sin(y(t) + h k_1^n)}{1 + h\cos(y(t) + h k_1^n)}$$