Problem 1

True/False:

1. The rate of convergence of $\cos(2h) + 2h\sin(h)$ as $h \to 0$ is $O(h^2)$
2. If $g(x)$ is a continuous function on the real line with $g(1) = -3$ and $g(2) = 1$, then g has at least one fixed point

1) F.

$$\cos(2h) = 1 - \frac{(2h)^2}{2!} + \frac{(2h)^4}{4!} - \cdots$$

$$\sin(h) = h - \frac{h^3}{3!} + \frac{h^5}{5!} - \cdots$$

$$\cos(2h) + 2h\sin(h) = 1 + O(h^4)$$

2) F. Fixed pt: $x = g(x)$,

IVT can't be applied here.

Problem 2

Consider $f(x) = (x - \cos(x))^3$.

1. Find starting point a, b such that the bisection method is guaranteed to converge. What is the order of convergence?
2. Use Newton's method to find an iteration $x_{n+1} = g(x_n)$ to find a root of $f(x)$

1) If $a = 0$, $f(0) = (-1)^3 = -1$

$b = \frac{\pi}{2}$, $f(\frac{\pi}{2}) = (\frac{\pi}{2} - \cos(\frac{\pi}{2}))^3 > 0$

So $[a, \frac{\pi}{2}]$ works.

2) $$x_{n+1} = x_n - \frac{[x - \cos(x_n)]^3}{3[x_n - \cos(x_n)][1 + \sin(x_n)]}$$

Problem 3

Consider the fixed point iteration $p_{n+1} = g(p_n)$ where $g(x) = \frac{1}{1+e^x}$:

1. Show that p_n converges to a unique fixed point for any initial guess $p_0 \in \mathbb{R}$
2. Find some other fixed point iteration $p_{n+1} = h(p_n)$ which converges to the same point but does so quadratically. Justify why your new fixed point iteration is quadratically convergent

1) $g'(x) = e^x \cdot \frac{-1}{(1+e^x)^2} < 0 \quad \forall x \in \mathbb{R}$

$\lim_{x \to -\infty} g(x) = 1$

$\lim_{x \to \infty} g(x) = 0$

$\rightarrow |g'(x)| < 1 \quad \forall x \in \mathbb{R}$, so this will always converge

2) $g(x) = x \rightarrow f(x) = 0$,

$f(x) = x - g(x)$

$\Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{1 - g'(x_n)}$

Problem 4

Let $\alpha > 1$ and $\lambda > 0$. Consider the following sequence:

$$p_k = \lambda^{\alpha^k},\ k = 1, 2, 3, \ldots$$

1. Find conditions under which $\lim_{k\to\infty} p_k = 0$
2. In the case where the limit is zero, find the order of convergence of this sequence

1) As $\alpha > 1$, $\alpha^k > 1$

$\Rightarrow$ If $|\lambda| < 1$, $\lim_{k\to\infty} p_k = 0$

2) $$\lim_{k\to\infty} \frac{|p_{k+1} - p|}{|p_k - p|^\beta} = \lim_{k\to\infty} \frac{\lambda^{\alpha^{k+1}}}{\lambda^{\alpha^k \cdot \beta}}$$

$$= \lim_{k\to\infty} \lambda^{\alpha^{k+1} - \beta\alpha^k}, \quad |\lambda| < 1$$

$$= \begin{cases} 0, & \beta < \alpha \\ 1, & \beta = \alpha \\ \infty, & \beta > \alpha \end{cases}, \quad \text{order is } \alpha$$

Problem 5

Find a polynomial that agrees with the function $f(x) = \sqrt{x}$ and its first derivative at $x =$ [redacted], 4. If the polynomial is used to approximate $f(2.5)$, what is the error upper bounded by?

$$f'(x) = \frac{1}{2} \cdot \frac{1}{\sqrt{x}}$$

$f(1) = 1 \quad f(4) = 2$

$f'(1) = \frac{1}{2} \quad f'(4) = \frac{1}{4}$

	f	$f(\cdot,\cdot)$	$f(\cdot,\cdot,\cdot)$	$f(\cdot,\cdot,\cdot,\cdot)$
1	1			
1	1	$\frac{1}{2}$		
4	2	$\frac{1}{3}$	$\frac{-1/6}{3} = -\frac{1}{18}$	
4	2	$\frac{1}{4}$	$\frac{-1/12}{3} = -\frac{1}{36}$	$\frac{1/36}{3} = \frac{1}{108}$

$$\Rightarrow p(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{12}(x-1)^2 (x-4)$$

Error: $\dfrac{(x-1)^2 (x-4)^2}{4!} f^{(4)}(\xi)$

$f^{(4)}(x) = \dfrac{-15}{16x^{7/2}}$, decreasing, so max at $x=1$

$$\Rightarrow |\text{error}| \le \frac{(2.5-1)^2 (2.5-4)^2}{4!} \cdot \frac{15}{16}$$