Problem 1

True/False:

1. The rate of convergence of $\cos(2h) + 2h\sin(h)$ as $h \to 0$ is $O(h^2)$
2. If $g(x)$ is a continuous function on the real line with $g(1) = -3$ and $g(2) = 1$, then g has at least one fixed point

Problem 2

Consider $f(x) = (x - \cos(x))^3$.

1. Find starting point a, b such that the bisection method is guaranteed to converge. What is the order of convergence?
2. Use Newton's method to find an iteration $x_{n+1} = g(x_n)$ to find a root of $f(x)$

Problem 3

Consider the fixed point iteration $p_{n+1} = g(p_n)$ where $g(x) = \frac{1}{1+e^x}$:

1. Show that p_n converges to a unique fixed point for any initial guess $p_0 \in \mathbb{R}$
2. Find some other fixed point iteration $p_{n+1} = h(p_n)$ which converges to the same point but does so quadratically. Justify why your new fixed point iteration is quadratically convergent

Problem 4

Let $\alpha > 1$ and $\lambda > 0$. Consider the following sequence:

$$p_k = \lambda^{\alpha^k},\ k = 1, 2, 3, ...$$

1. Find conditions under which $\lim_{k\to\infty} p_k = 0$
2. In the case where the limit is zero, find the order of convergence of this sequence

Problem 5

Find a polynomial that agrees with the function $f(x) = \sqrt{x}$ and its first derivative at $x = 1, 4$. If the polynomial is used to approximate $f(2.5)$, what is the error upper bounded by?