Matrix Multiply (20 Points)

In this problem you will implement matrix-matrix multiply two ways, using iteration and recursion. For all the problems we will assume that the matrices are of size $2^k \times 2^k$ for some integer $k \geq 0$. Recall also that matrix-matrix multiplication is defined as

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

**Problem 1 (7 points)**

Write a Julia function that takes in two input matrices $A$ and $B$ and returns the product $C = AB$ using for loops. You may only do scalar multiplication and addition, and may not use any built in functionality that performs matrix-matrix or matrix-vector multiplication.

```julia
function matmatmul(A, B)
    n = size(A)[1]
    C = zeros((n, n))
    for i=1:n
        for j=1:n
            for k=1:n
                C[i,j] += A[i,k] * B[k,j]
            end
        end
    end
    return C
end
```

Now you will implement matrix-matrix multiply recursively. Do this by splitting the matrices as

$$C = AB = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where $A$ is size $n \times n$ and each of the four submatrices is size $\frac{n}{2} \times \frac{n}{2}$. 

```julia
function matmatmul_recursive(A, B)
    if size(A)[1] == 1
    else
        A11, A12, A21, A22 = split(A, Diagonal)
        B11, B12, B21, B22 = split(B, Diagonal)
        C11, C12, C21, C22 = matmatmul_recursive(A11, B11), matmatmul_recursive(A11, B12), matmatmul_recursive(A21, B11), matmatmul_recursive(A21, B12)
        return [C11 + C12, C21 + C22, C11 + C21, C12 + C22]
    end
end
```
Problem 2 - 2 Points
What is the base case?

When \( n=1 \), then \( C=AB \) is just scalar multiplication.

Problem 3 - 3 Points
Write out the expression for each submatrix \( C_{ij} \) in terms of the submatrices \( A_{ij}, B_{ij} \).

\[
C_{11} = A_{11} B_{11} + A_{12} B_{21} \\
C_{12} = A_{11} B_{12} + A_{12} B_{22} \\
C_{21} = A_{21} B_{11} + A_{22} B_{21} \\
C_{22} = A_{21} C_{12} + A_{22} C_{22}
\]

Problem 4 - 8 Points
Write Julia code to implement matrix-matrix multiply recursively using this splitting method.

```julia
function recur(C, A, B)
    n = size(A)[1]
    C = zeros(n, n)
    if n==1
        return A*B
    end

    # Generate submatrices
    A11 = A[1:n/2, 1:n/2]; B11 = B[1:n/2, 1:n/2]
    A12 = A[1:n/2, n/2+1:end]; B12 = B[1:n/2, n/2+1:end]
    A21 = A[n/2+1:end, 1:n/2]; B21 = B[n/2+1:end, 1:n/2]
    A22 = A[n/2+1:end, n/2+1:end]; B22 = B[n/2+1:end, n/2+1:end]

    # Recursive calls
    C[1:n/2, 1:n/2] = recur(A11, B11) + recur(A12, B21)
    C[1:n/2, n/2+1:end] = recur(A11, B12) + recur(A12, B22)
    C[n/2+1:end, 1:n/2] = recur(A21, B11) + recur(A22, B21)
    C[n/2+1:end, n/2+1:end] = recur(A21, B12) + recur(A22, B22)

    return C
end
```