Producing Ricci flows by singular Ricci flows

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Structure of Talk:

- **Part I** Introduction
- **Part II** Singular Ricci flow
- **Part III** Generalized singular Ricci flow
- **Part IV** Proof of the main theorem
Part I Introduction
Ricci flow equation:

\[
\frac{d}{dt} g(t) = -2\text{Ric}(g(t))
\] (0.1)

**Theorem (Hamilton)**

Let \( M \) be a compact \( n \)-dimensional manifold, there exists a short time Ricci flow starting from \( M \).

Compact RF preserves \( \text{Ric} \geq 0 \) in 3d. Curvature blows up in finite time.

**Theorem (Shi)**

Let \( M \) be a complete \( n \)-dimensional manifold with bounded curvature, there exists a short time complete Ricci flow starting from \( M \).

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Theorem (Simon, Topping, 2017)

Let $(M, g)$ be a 3d complete manifold. Suppose $Vol_g(B_g(x, 1)) \geq v_0$ (non-collapsing) and $\text{Ric} \geq -1$ everywhere (curvature lower bound). Then there exists a Ricci flow $(M, g(t)), t \in [0, T]$ with $g(0) = g$.

Idea: Take an exhaustion of $M$ by compact subsets $U_i$. For each $U_i$, construct a local Ricci flow $(U_i, g_i(t)), t \in [0, T]$, by running Shi’s Ricci flow inductively. Take a limit of $(U_i, g_i(t))$ to get a Ricci flow $(M, g(t))$.

Two key curvature estimates:

- $|Rm|_{g_i(t)} \leq \frac{C}{t}$. Suppose this is not true, there is a sequence of Ricci flows converging to a $\kappa$-solution. The non-collapsing assumption implies the asymptotic volume ratio is non-zero, contradiction.
- $\text{Ric} \geq -C$, obtained by a bootstrap argument.
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Some invariant curvature conditions:

1. non-negative curvature operator;
2. non-negative complex sectional curvature (weakly $PIC_2$);
3. 2-non-negative curvature operator ($\text{Ric} \geq 0$ in 3d);
4. weakly $PIC_1$;

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In arbitrary dimension, there exists a complete Ricci flow starting from a complete manifold with non-negative complex sectional curvature.

Idea: take \((M_i, p_i) \rightarrow (M, p)\), where \(M_i\) is compact and has non-negative complex sectional curvature. So the same holds for \((M_i, g_i(t)), t \in [0, T_i]\), and \(\lim_{t \uparrow T_i} Vol_t(M_i) = 0\). By Petrunin’s result, \(\int_{B_t(p_i, 1)} R \, dvol \leq C\), it implies \(T_i \geq T\) for all \(i\). Then take a convergent subsequence of \((M_i, g_i(t)), t \in [0, T]\).

Note, in 3d, complex sec \(\geq 0 \Leftrightarrow\) sec \(\geq 0 \Rightarrow\) Ric \(\geq 0\).

**Question**: is Ric \(\geq 0\) in 3d sufficient to run a Ricci flow?
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A conjecture by Topping

Given a 3d complete Riemannian manifold \((M, g)\) with \(\text{Ric} \geq 0\), there is a smooth continuation by Ricci flow.

Main theorem (L, 2020)

Given a 3d complete Riemannian manifold \((M, g)\) with \(\text{Ric} \geq 0\), there is a smooth Ricci flow \((M, g(t)), t \in [0, T_{\max})\), starting out from \((M, g)\). Moreover, if \(T_{\max} < \infty\), then \(\limsup_{t \uparrow T_{\max}} |\text{Rm}|(x, t) = \infty\) for all \(x \in M\).

E.g. \(T_{\max} < \infty\): the standard solution, \(S^2 \times \mathbb{R}\); \(T_{\max} = \infty\): Bryant soliton

Strategy to construct the flow:

- run a generalized singular Ricci flow \(\mathcal{M}\);
- show \(\text{Ric} \geq 0\) holds on \(\mathcal{M}\);
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<thead>
<tr>
<th>Example</th>
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Part II  Singular Ricci flow
A Ricci flow spacetime \((\mathcal{M}, g(t))\) is the following:

- \(\mathcal{M}\) is a 4-manifold with boundary.
- time function \(t : \mathcal{M} \to [0, T]\), time-t-slice \(\mathcal{M}_t\), and \(\mathcal{M}_0 = \partial \mathcal{M}\).
- \(\partial_t\) is a smooth vector field in \(\mathcal{M}\), \(\partial_t t = 1\).
- \(g\) is a metric on \(\ker(dt)\).
- \(\mathcal{L}_{\partial_t} g = -2Ric(g(t))\).

Canonical neighborhood assumption (CNA): Let \(M\) be a 3d manifold. We say that the \(\epsilon\)-CNA holds at \(x \in M\), if \((M, x)\) is \(\epsilon\)-close to a \(\kappa\)-solution at scale \(R(x)^{-1/2}\).
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**$\epsilon$-neck:** A region that is $\epsilon$-close to $S^2 \times \mathbb{R}$ under rescaling.

**Strong $\epsilon$-neck:** A spacetime region that is $\epsilon$-close to $(S^2 \times \mathbb{R}, g(t))$ for $t \in [-1, 0]$ under rescaling.

**Gradient estimates:** If $\epsilon$-CNA holds at $x$, then

$$|\nabla R^{1/2}|(x) \leq C, \quad |\partial_t R^{-1}|(x) \leq C.$$  \hspace{1cm} (0.2)

We say a Ricci flow spacetime $\mathcal{M}$ is 0-complete (resp. backward 0-complete) if for any smooth curve $\gamma : [0, s_0) \to \mathcal{M}$ that satisfies $\inf_{[0,s_0)} R(\gamma(s)) < \infty$ and one of the following, then $\lim_{s \to s_0} \gamma(s)$ exists:

- $\gamma([0, s_0))$ is contained in a time-slice $\mathcal{M}_t$, and has finite length with respect to the horizontal metric in $\mathcal{M}_t$, or
- $\gamma$ is the integral curve of $-\partial_t$, or $\partial_t$ (resp. only $-\partial_t$).
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Singular Ricci flow

Theorem (Kleiner, Lott, 2014)

Let \((M, g)\) be a 3d compact manifold, then there exists a singular Ricci flow starting from \(M\), which is a Ricci flow spacetime that satisfies

- \(\mathcal{M}_0 = M\) is compact;
- \(\mathcal{M}\) is 0-complete;
- For any \(x \in \mathcal{M}\), \(t(x) \leq T\), if \(R(x) \geq r^{-2}(T)\), then the \(\epsilon\)-CNA holds at \(x\).

Theorem: For any \(x_0 \in M\), suppose \(x_0\) survives until \(t_0 > 0\), then

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\mathcal{N} := \bigcup_{t=0}^{t_0} \bigcup_{A>0} B_t(x_0(t), A)
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Let $(\mathcal{M}, g(t))$ be a singular Ricci flow with normalized initial condition, $x_0 \in \mathcal{M}$, $t(x_0) = t_0$. Suppose $|Rm| \leq r_0^{-2}$ in $P_0 := P(x_0, t_0, r_0, -r_0^2)$, then

Theorem (Heat kernel)

Then there is a solution $u \geq 0$ to $(-\partial_t - \Delta + R)u = 0$, $u$ is a $\delta$-function at $x_0$, and $C_m = C_m(r_0)$, such that

$$uR^m \leq C_m \quad \text{in} \quad \mathcal{M}_{t < t_0 - P_0}$$

(0.4)

Step 1 (construct $u$): Let $\mathcal{M}_i \to \mathcal{M}$ be a sequence of Ricci flow with surgeries. Define $u_i$ on $\mathcal{M}_i$ by integrating with the ordinary heat kernels. Then $u_i \to u$. 
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Step 2 (a vanishing theorem): Studying the solution $u \geq 0$ to 
$(-\partial_t - \Delta + R)u = 0$ in a non-compact $\kappa$-solution on $[0, T_{\text{max}})$.

For example, in a Bryant soliton: If $uR^m \leq C$, then $u \equiv 0$.

Step 3 (a semi-local maximum principle): For any $x_1$ with sufficiently large $R$, there is $x_2$ with $t(x_2) \geq t(x_1)$ such that

$$\begin{cases} 
    uR^m(x_2) \geq (1 + \epsilon_m)uR^m(x_1), \\
    u(x_2) \geq (1 + \epsilon_m)u(x_1).
\end{cases} \quad (0.5)$$

Prove $(0.5)$ by a limiting argument: Suppose it is violated in a sequence 
$(M_i, x_i, u_i)$, with $R(x_i) \to \infty$. Then rescale each flow by $R(x_i)$, and rescale $u_i$ such that $u_i(x_i) = 1$. Then

$$(M_i, x_i, u_i) \to (g_\infty(t), x_\infty, u_\infty), \quad (0.6)$$

where $g_\infty(t)$ is a non-compact $\kappa$-solution defined on $[0, T_{\text{max}})$. By step 2 we get a contradiction. Prove the theorem by using $(0.5)$ repeatedly.
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\[
(M_i, x_i, u_i) \to (g_\infty(t), x_\infty, u_\infty),
\] (0.6)

where \( g_\infty(t) \) is a non-compact \( \kappa \)-solution defined on \([0, T_{\text{max}})\). By step 2 we get a contradiction. Prove the theorem by using (0.5) repeatedly.
Corollary: $\int_{\mathcal{M}_t} u \, d_t \text{vol} = 1$ for all $t \in [0, t_0)$.

Corollary: Pseudolocality theorem for singular Ricci flows.

**Canonical neighborhood theorem**

Let $(\mathcal{M}, g, x_0)$ be a singular Ricci flow, $x_0 \in \mathcal{M}_0$. Suppose $|\text{Rm}| \leq 1$ and $\text{vol}(B_1(x_0, 1)) \geq A^{-1}$ on $P(x_0, 0; 1, 1)$. Then there exists $r(A) > 0$ such that the $\epsilon$-CNA holds in $B_1(x_0, A)$ at scales less than $r(A)$.

Remark: Unlike the case of Ricci flow with surgeries, there is no need to assume that the initial condition of $\mathcal{M}$ is normalized, thanks to the 'zero-surgery scale' of the singular Ricci flow.
Corollary: \( \int_{\mathcal{M}_t} u \, dt \, vol = 1 \) for all \( t \in [0, t_0) \).

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Part III Generalized singular Ricci flow
Theorem (L, 2020)

Let \((M, g)\) be a 3d complete manifold (with possibly unbounded curvature). Then there exists a \textbf{generalized singular Ricci flow} \(\mathcal{M}\) starting from \((M, g)\), which is a Ricci flow spacetime that satisfies:

- \(\mathcal{M}_0 = M\) is complete;
- \(\mathcal{M}\) is 0-complete;
- For any fixed \(x_0 \in \mathcal{M}\), \(t(x_0) = t_0\), \(\epsilon\)-CNA holds on \(B_{t_0}(x_0, A)\) at scales \((0, r(A))\).
Proof of the Theorem:

Pick \( x_0 \in M \), and a sequence of compact manifolds

\[
(M_i, x_{0i}) \longrightarrow (M, x_0).
\]  

(0.7)

Let \( M_i \) be singular Ricci flows with \( M_{i,0} = M_i \).

By the pseudolocality theorem,
\[
x \in B_t(x_{0i}, A), \ t \in [0, t(A)] \Rightarrow |Rm|(x) \leq C(A).
\]

Take \( T = t(10) \). By the canonical neighborhood theorem,
\[
x \in B_t(x_{0i}, A), \ t \in [t(A), T] \Rightarrow \epsilon\text{-CNA holds if } |Rm| \geq r(A)^{-2}.
\]

In summary, by decreasing \( r(A) \), we have
\[
x \in B_t(x_{0i}, A), \ t \in [0, T] \Rightarrow \epsilon\text{-CNA holds if } |Rm| \geq r(A)^{-2}.
\]

Therefore, for any fixed \( A \), \( B_t(x_{0i}, A) \) is uniformly totally bounded.
Proof of the Theorem:

Pick $x_0 \in M$, and a sequence of compact manifolds

$$ (M_i, x_{0i}) \longrightarrow (M, x_0). \quad (0.7) $$

Let $\mathcal{M}_i$ be singular Ricci flows with $\mathcal{M}_{i,0} = M_i$.

By the pseudolocality theorem,

$$ x \in B_t(x_{0i}, A), \; t \in [0, t(A)] \Rightarrow |\text{Rm}(x)| \leq C(A). $$

Take $T = t(10)$. By the canonical neighborhood theorem,

$$ x \in B_t(x_{0i}, A), \; t \in [t(A), T] \Rightarrow \epsilon\text{-CNA holds if } |\text{Rm}| \geq r(A)^{-2}. $$

In summary, by decreasing $r(A)$, we have

$$ x \in B_t(x_{0i}, A), \; t \in [0, T] \Rightarrow \epsilon\text{-CNA holds if } |\text{Rm}| \geq r(A)^{-2}. $$

Therefore, for any fixed $A$, $B_t(x_{0i}, A)$ is uniformly totally bounded.
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Pick \( x_0 \in M \), and a sequence of compact manifolds

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In summary, by decreasing \( r(A) \), we have

\[
x \in B_t(x_{0i}, A), \ t \in [0, T] \Rightarrow \epsilon\text{-CNA holds if } |Rm| \geq r(A)^{-2}.
\]

Therefore, for any fixed \( A, B_t(x_{0i}, A) \) is uniformly totally bounded.
Proof of the Theorem:

Pick $x_0 \in M$, and a sequence of compact manifolds

$$(M_i, x_{0i}) \rightarrow (M, x_0).$$

(0.7)

Let $M_i$ be singular Ricci flows with $M_{i,0} = M_i$.

By the pseudolocality theorem,

$x \in B_t(x_{0i}, A), \ t \in [0, t(A)] \Rightarrow |Rm|(x) \leq C(A)$.

Take $T = t(10)$. By the canonical neighborhood theorem,

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Proof of the Theorem:

Pick $x_0 \in M$, and a sequence of compact manifolds

$$\left( M_i, x_{0,i} \right) \longrightarrow (M, x_0).$$ \hfill (0.7)

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Therefore, for any fixed $A$, $B_t(x_{0,i}, A)$ is uniformly totally bounded.
Let \( G_i = dt^2 + g_i(t) \), and \( d_i \) be the metric induced by \( G_i \).

Let \( P_i(A) := \bigcup_{t \in [0, T)} B_t(x_{0i}, A) \). Then \((P_i(A), d_i)\) is uniformly totally bounded. So

\[
(P_i(A), d_i) \xrightarrow{GH} (X(A), d_A). \tag{0.8}
\]

Let \( \mathcal{N}_i = \bigcup_{A > 0} P_i(A) \), then

\[
(\mathcal{N}_i, d_i, x_{0i}) \xrightarrow{pGH} (X, d, x_0). \tag{0.9}
\]

Let \( \mathcal{M} = \{\text{‘smooth points’ in } X\} \). By the gradient estimate, there is a smooth spacetime metric on \( \mathcal{M} \), \( t(\mathcal{M}) = [0, T) \), and

\[
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Generalized singular Ricci flow

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By taking $T$ maximal, we can assume that $x_0$ survives until its curvature blows up.

Moreover, we can show that $(\mathcal{M}, x_0)$ is backward 0-complete. By taking a 'union' of all such $(\mathcal{M}, x_0)$ we get a generalized singular Ricci flow.
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Part IV Proof of the main theorem
Proof of the main theorem

Lemma

Let \((M, g)\) be a complete 3-manifold with \(\text{Ric} \geq 0\) (resp. \(R \geq 0\)). Let \(\mathcal{M}\) be a generalized singular Ricci flow starting from \((M, g)\). Then \(\text{Ric} \geq 0\) (resp. \(R \geq 0\)) on \(\mathcal{M}\).

To show \(R \geq 0\) is preserved, note

- In each \(\mathcal{M}_t\), \(R\) is positive in the high curvature regions. So \(R_{\min} < 0\) is achieved at some point.
- \(\bigcup_{t \in [0, T)} \bigcup_{A > 0} B_t(x_0(t), A)\) is backward 0-complete. It guarantees

\[
\liminf_{t \searrow t_0} R_{\min}(t) \geq R_{\min}(t_0). \tag{0.11}
\]

Then apply maximum principle.

We can show \(\text{Ric} \geq 0\) in a similar way.
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Main theorem (L, 2020)
Given a 3d complete Riemannian manifold \((M, g)\) with \(\text{Ric} \geq 0\), there is a smooth Ricci flow \((M, g(t)), t \in [0, T_{\text{max}}]\), starting out from \((M, g)\). Moreover, if \(T_{\text{max}} < \infty\), then \(\limsup_{t \uparrow T_{\text{max}}} |\text{Rm}|(x, t) = \infty\) for all \(x \in M\).

Proof: Let \((M, g(t))\) be a generalized singular Ricci flow starting from \(M\). Let \(x_0 \in M\). Suppose \(x_0\) survives until \(T > 0\). We claim that \(M_t\) is complete for all \(t \in [0, T]\).

Suppose not, then for some \(t, A > 0\) there is a minimizing geodesic \(\gamma : [0, 1) \to B_t(x_0, A)\) such that \(\lim_{s \to 1} R(\gamma(s)) = \infty\), and \(\gamma(s)\) is center of strong \(\epsilon\)-necks for all \(s\) close to 1.
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Proof of the main theorem

Let $X = \{p\} \cup B_t(x_0, A)$ be the one-point completion, and take a blow-up limit of $X$ at $p$,

$$\lambda X \xrightarrow{GH} X_\infty, \text{ as } \lambda \to \infty. \quad (0.12)$$

Then by $\text{Ric} \geq 0$, we can show $X_\infty$ is a smooth cone.

Since for any $x \in X_\infty$, $x$ is the center of a strong $2\epsilon$-neck, $X_\infty$ is flat. However, by the gradient estimate on $X$,

$$|\nabla R^{-\frac{1}{2}}| \leq C \Rightarrow R^{-\frac{1}{2}}(x) \leq C d(x, p). \quad (0.13)$$

So $X_\infty$ is not flat, a contradiction. So $M_t$ is complete for all $t \in [0, T]$.

Since $\text{Ric} \geq 0$, we have $d_t(x, x_0) \leq d_0(x, x_0)$ for any $x \in M$. So $M$ survives until $T$, and $M \times [0, T] \subset M$ is a smooth Ricci flow.
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Thanks for your listening!