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# Ricci flow under local almost non-negative curvature conditions

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ABSTRACT

We find a local solution to the Ricci flow equation under a negative lower bound for many known curvature conditions. Under a non-collapsing assumption, the flow exists for a uniform amount of time, during which the curvature stays bounded below by a controllable negative number. The curvature conditions we consider include 2-non-negative and weakly $\mathrm{PIC}_1$ cases, of which the results are new. We complete the discussion of the almost preservation problem by Bamler–Cabezas-Rivas–Wilking, and the 2-non-negative case generalizes a result in 3D by Simon–Topping to higher dimensions.

As an application, we use the local Ricci flow to smooth a metric space which is the limit of a sequence of manifolds with the almost non-negative curvature conditions, and show that this limit space is bi-Hölder homeomorphic to a smooth manifold.



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## 1. Introduction and main results

Ricci flow as introduced by Hamilton in [9], describes the evolution of a time-dependent family $g(t)_{\{t\in I\}}$ of Riemannian metrics on a manifold $M$:

$$\frac{\partial}{\partial t}g(t) = -2\mathrm{Ric}(g(t)).$$

Here $\mathrm{Ric}(g(t))$ denotes the Ricci curvature of the metric $g(t)$. Hamilton used Ricci flow to prove that a compact three-manifold admitting a Riemannian metric of positive Ricci curvature must be a spherical space form. Since then Ricci flow has been used to prove many conjectures including the most remarkable Poincaré and Geometrization Conjectures in dimension 3 by Perelman ([17–19]).

In general, Ricci flow tends to preserve some kind of positivity of curvatures. For example, positive scalar curvature is preserved in all dimensions. This follows from applying maximum principle to the evolution equation of scalar curvature, which is

$$\frac{\partial}{\partial t}\mathrm{scal} = \Delta\mathrm{scal} + 2|\mathrm{Ric}|^2.$$

By developing a maximum principle for tensors, Hamilton [9,10] proved that Ricci flow preserves the positivity of the Ricci tensor in dimension three and positivity of the curvature operator in all dimensions. H. Chen [5] also proved the preservation of 2-non-negative curvature. The invariance of weakly PIC was first shown in dimension four by Hamilton [12], and the general case was obtained independently by Brendle and Schoen [3] and by Nguyen [16]. The curvature conditions weakly $\mathrm{PIC}_1$ and $\mathrm{PIC}_2$ were in turn introduced by Brendle and Schoen in [3] and played a key role in their proof of the differentiable sphere theorem. Finally in the Kähler case, the condition of non-negative holomorphic bisectional curvature, which is a weaker condition than non-negative sectional curvature, is also preserved. This was shown by Bando [2] in dimension three and by Mok [15] in all dimensions. In [21] Shi generalized this result to the complete Kähler manifolds with bounded curvature.

In this paper, we study the preservation of almost non-negativity of curvature conditions. We say a quantity is almost non-negative when it has a negative lower bound. The almost non-negative case is less restrictive since it puts no constraints on the topology of the manifold. In [1], Bamler, Cabezas-Rivas, and Wilking studied the complete manifold with bounded curvature, which satisfies global non-collapsedness and almost non-negativity for some curvature conditions. They showed that under the assumption, a Ricci flow exists for a uniform amount of time, during which the curvature can be bounded below by a negative constant depending only on initial conditions. In the same paper, they also established some local results without the complete and curvature bound assumptions.

However, the local cases of almost 2-non-negative curvature and weakly $\mathrm{PIC}_1$ remained unsolved. We verify these two local cases in this paper. We use $\mathcal{C}$ to denote various

non-negative curvature conditions, and write $\mathrm{Rm} \in \mathcal{C}$ to indicate that the curvature operator Rm satisfies the corresponding curvature condition. Then $\mathrm{Rm}+C\mathrm{I} \in \mathcal{C}$ indicates the nonnegativity of $\mathrm{Rm} + C\mathrm{I}$, where I is the identity curvature operator whose scalar curvature is $n(n-1)$. Under this notation, our main theorem can be stated as below:

**Theorem 1.1.** *Given $n \in \mathbb{N}$, $\alpha_0 \in (0,1]$ and $v_0 > 0$, there exist positive constants $\tau = \tau(n, v_0, \alpha_0)$ and $C = C(n, v_0)$ such that the following holds: Let $(M^n, g_0)$ be a Riemannian manifold (not necessarily complete) and consider one of the following curvature conditions $\mathcal{C}$:*

1. *non-negative curvature operator;*
2. *2-non-negative curvature operator*
   *(i.e. the sum of the lowest two eigenvalues of the curvature operator is non-negative);*
3. *weakly $PIC_2$*
   *(i.e. non-negative complex sectional curvature, meaning that taking the cartesian product with $\mathbb{R}^2$ produces a non-negative isotropic curvature operator);*
4. *weakly $PIC_1$*
   *(i.e. taking the cartesian product with $\mathbb{R}$ produces a non-negative isotropic curvature operator).*

*Suppose $B_{g_0}(x_0, s_0) \subset\subset M$ for some $x_0 \in M$ and $s_0 > 4$ such that*

$$\begin{cases} Rm_g + \alpha_0 I \in \mathcal{C} & \text{on} \quad B_{g_0}(x_0, s_0) \\ Vol_{g_0} B_{g_0}(x, 1) \geq v_0 > 0 & \text{for all} \quad x \in B_{g_0}(x_0, s_0 - 1)\ . \end{cases} \tag{1.1}$$

*Then there exists a Ricci flow $g(t)$ defined for $t \in [0, \tau]$ on $B_{g_0}(x_0, s_0 - 2)$, with $g(0) = g_0$, such that for all $t \in [0, \tau]$,*

$$\begin{cases} |Rm|_{g(t)} \leq \dfrac{C}{t} & \text{on} \quad B_{g_0}(x_0, s_0 - 2) \\ Rm_{g(t)} + C\alpha_0 I \in \mathcal{C}. \end{cases} \tag{1.2}$$

The results of the first and third conditions above were obtained in [1]. In dimensional three, 2-non-negative curvature has the same meaning as non-negative Ricci curvature, where the result was obtained by Simon and Topping in [23] and [24].

For each curvature condition $\mathcal{C}$, we define $\ell(x) \geq 0$ to be the smallest number such that $\mathrm{Rm}_g(x) + \ell(x)\,\mathrm{I} \in \mathcal{C}$. Then in each case the bound $\ell \leq 1$ implies a lower bound on the Ricci curvature. We also observe that each curvature condition implies weakly $\mathrm{PIC}_1$. The method we use in the paper allows a uniform treatment of all curvature conditions that imply a lower bound for Ricci curvature and weakly $\mathrm{PIC}_1$.

As an application we have the following global existence result on complete manifolds with possibly unbounded curvature. It extends the corresponding results in [1] to the 2-non-negative and weakly $\mathrm{PIC}_1$ cases.

**Corollary 1.2.** *Given $n \in \mathbb{N}, \alpha_0 \in (0,1]$ and $v_0 > 0$, there exist positive constants $C = C(n, v_0)$ and $\tau = \tau(n, v_0, \alpha_0)$ such that the following holds: Let $\mathcal{C}$ be any curvature conditions listed in Theorem 1.1, and $(M^n, g)$ be any complete Riemannian manifold (with possibly unbounded curvature) such that*

$$\begin{cases} Rm_g + \alpha_0 I \in \mathcal{C} \\ Vol_g B_g(p,1) \geq v_0 \quad \text{for all } p \in M. \end{cases} \tag{1.3}$$

*Then there exists a complete Ricci flow $(M, g(t))_{t\in(0,\tau]}$ with $g(0) = g$ and so that*

$$\begin{cases} Rm_{g(t)} + C\alpha_0 I \in \mathcal{C} \quad \text{for all } t \in (0,\tau] \text{ throughout } M \\ |Rm|_{g(t)} \leq \dfrac{C}{t}. \end{cases} \tag{1.4}$$

To prove the corollary we apply the local Ricci flow in Theorem 1.1 to a sequence of larger and larger balls on the complete manifold. Thanks to the curvature decay estimate $|\mathrm{Rm}| \leq \frac{C}{t}$ in (1.2), we can then take a convergent subsequence and get a globally defined flow.

Another application is the following smoothing result for singular limit spaces of sequences of manifolds with lower curvature bounds, which asserts the limit space is bi-Hölder homeomorphic to a smooth manifold.

**Corollary 1.3.** *Given $n \in \mathbb{N}$, $\alpha_0, v_0 > 0$. Let $\mathcal{C}$ be any curvature conditions listed in Theorem 1.1, and $(M_i^n, g_i)$ be a sequence of complete Riemannian manifolds such that for all $i$, we have*

$$\begin{cases} Rm_{g_i} + \alpha_0 I \in \mathcal{C} \quad \text{throughout } M_i \\ Vol_{g_i} B_{g_i}(x,1) \geq v_0 \quad \text{for all } x \in M_i \end{cases} \tag{1.5}$$

*Then there exists a smooth manifold $M$, a point $x_\infty \in M$, and a continuous distance metric $d_0$ on $M$ such that for some points $x_i \in M_i$, a subsequence of $(M_i, d_{g_i}, x_i)$ converges in the pointed Gromov–Hausdorff distance sense to $(M, d_0, x_\infty)$. Furthermore, the metric space $(M, d_0)$ is bi-Hölder homeomorphic to the smooth manifold $M$ equipped with any smooth metric.*

We give the proofs of Corollary 1.2 and 1.3 in Section 8. We mention here that with some careful local distance distortion arguments, the same conclusion in Corollary 1.3 holds provided noncollapsedness of only one ball centered at a point. For detailed proof of this, we refer to [24] where the argument is done for Ricci curvature and carries over to our case.

Finally, we sketch the proof of Theorem 1.1 under some additional assumptions. That is, assuming (1.1) holds globally and a short time Ricci flow exists up to a uniform time

$T < 1$, during which $|\mathrm{Rm}| \leq \frac{C}{t}$ holds, we want to show $\mathrm{Rm}_{g(t)} + C\alpha_0 \mathrm{I} \in \mathcal{C}$ for all $t$. We define $\ell(x,t)$ by

$$\ell(x,t) := \inf\{\varepsilon \in [0,\infty) | \mathrm{Rm}_{g(t)}(x) + \varepsilon \mathrm{I} \in \mathcal{C}\}. \tag{1.6}$$

Then it's equivalent to show $\ell(\cdot,t) \leq C\alpha_0$ for all $t$. By [1, Proposition 2.2], $\ell$ satisfies an evolution inequality of the form

$$\frac{\partial}{\partial t}\ell \leq \Delta \ell + \mathrm{scal}\,\ell + C(n)\ell^2 \tag{1.7}$$

in the barrier sense for some dimensional constant $C(n)$. Assuming $\ell(x,t) \leq 1$, then by the maximum principle, $\ell(\cdot,t) \leq e^{C(n)t}h$ on $M \times [0,t)$, where $h$ solves

$$\frac{\partial}{\partial t}h = \Delta h + \mathrm{scal}\,h, \quad h(\cdot,0) = \ell(\cdot,0). \tag{1.8}$$

We can express this solution as

$$h(x,t) = \int_M G(x,t;y,0)\,\ell(y,0)\,d_0y, \tag{1.9}$$

where $G(\cdot,\cdot;y,s)$ satisfies

$$(\frac{\partial}{\partial t} - \Delta_{x,t} - \mathrm{scal}_{g(t)})G(x,t;y,s) = 0 \quad \text{and} \quad \lim_{t\searrow s} G(x,t;y,s) = \delta_y(x). \tag{1.10}$$

We say $G(\cdot,\cdot;y,s)$ is the heat kernel of equation (1.10). It can be shown with the bound $|\mathrm{Rm}|_{g(t)} \leq \frac{C}{t}$ that $G(x,t;y,s)$ has the following Gaussian upper bound

$$G(x,t;y,s) \leq \frac{C}{(t-s)^{\frac{n}{2}}} exp\left(-\frac{d_s^2(x,y)}{C(t-s)}\right), \tag{1.11}$$

substituting which into (1.9) we get

$$\begin{aligned}\ell(x,t) \leq e^{C(n)}h(x,t) &\leq \sup_{y\in M}\ell(y,0)\cdot\frac{C}{t^{\frac{n}{2}}}\int_M exp\left(-\frac{d_0^2(x,y)}{Ct}\right)d_0y \\ &\leq C\sup_{y\in M}\ell(y,0).\end{aligned} \tag{1.12}$$

To prove Theorem 1.1 by adapting the above argument, we need to overcome the difficulties caused by the lack of those additional assumptions. To construct a local Ricci flow, we use an extension method which was introduced in [13] and [24]. The process starts by doing a conformal change to the initial metric, making it a complete metric and leaving it unchanged on $B_{g(0)}(x_0,r_1)$ for some $0 < r_1 < r_0 = s_0$. Then by the following

doubling time estimate of Shi in [20], we can then run a complete Ricci flow up to a short time $t_1$.

**Lemma 1.4.** *(Doubling time estimate) Let $(M^n, g(0))$ be a complete manifold with bounded curvature $|Rm|_{g(0)} \leq K$, then there exists a complete Ricci flow $(M^n, g(t))$ such that*

$$|Rm|_{g(t)} \leq 2K \tag{1.13}$$

*for all $0 \leq t \leq \frac{1}{16K}$.*

Of course $t_1$ is uncontrolled and may depend on specific manifold due to the lack of a uniform curvature bound. Next we do another conformal change to complete the metric at $t_1$, leaving it unchanged on $B_{g(0)}(x_0, r_2)$ for some $0 < r_2 < r_1$. Then using the doubling time estimate again, we have another complete Ricci flow from $t_1$ to $t_2$. Repeating the process, we obtain some successive complete Ricci flow pieces $(\{M_i\}_{i=1}^m, \{g_i(t)\}_{i=1}^m)$, with each $M_i$ containing $B_{g(0)}(x_0, r_i)$. Restricting all the $g_i(t)$ on $B_{g(0)}(x_0, r_m)$, we thus obtain a smooth local Ricci flow $g(t)$ defined for all $t \in [0, t_m]$. The inductive construction is carried out in Section 6.

In particular, the curvature decay $|\mathrm{Rm}|_{g(t)} \leq \frac{C}{t}$ in (1.2) together with the doubling time estimate enable us to choose $t_{i+1} = t_i(1 + \frac{1}{16C})$ for each $i$. To verify $|\mathrm{Rm}|_{g(t)} \leq \frac{C}{t}$ after each extension step, we use the curvature decay lemma in Section 3, which ensures the existence of $C$ under the assumption of a local upper bound of $\ell(\cdot, t)$.

For the verification of $\ell(\cdot, t) \leq C\alpha_0$ in (1.2), we perform a new local integration estimate, in which we use a generalized heat kernel. We know the standard heat kernel $G(x, t; y, s)$ on a complete Ricci flow satisfies the following reproduction formula for all $\mu < s < t$

$$\int G(x, t; y, s)\, G(y, s; z, \mu)\, d_s y = G(x, t; z, \mu). \tag{1.14}$$

The standard heat kernel $G(x, t; y, s)$ is well defined by equation (1.10) for all $(x, t)$ and $(y, s)$ in a same complete Ricci flow piece $(M_i, g_i(t))$ coming from the above inductive construction. In section 5, we use equation (1.14) inductively to make sense of $G(x, t; y, s)$ for $(x, t)$ and $(y, s)$ in different pieces and thus obtain a generalized heat kernel whose definition domain is on the whole $(\{M_i\}_{i=1}^m, \{g_i(t)\}_{i=1}^m)$ and has a Gaussian upper bound.

## 2. Preliminaries

### *2.1. Local distance distortion estimates*

We need the following distance distortion estimates, which are originally due to Hamilton [11] and Perelman [17], and phrased and improved in [24]. These estimates ensure the

distance between two points won't expand or shrink too soon when assuming $\mathrm{Ric} \geq -K$ or $\mathrm{Ric} \leq \frac{C}{t}$, respectively.

**Lemma 2.1.** *(Expanding Lemma). Given $T, K, R > 0$ and $n \in \mathbb{N}$. Let $(M^n, g(t))$ be a Ricci flow for $t \in [-T, 0]$. Suppose for some $x_0 \in M$ we have $B_{g(0)}(x_0, R) \subset\subset M$ and $Ric_{g(t)} \geq -K$ on $B_{g(0)}(x_0, R) \cap B_{g(t)}(x_0, Re^{Kt})$ for each $t \in [-T, 0]$.*

*Then for all $t \in [-T, 0]$,*

$$B_{g(0)}(x_0, R) \supset B_{g(t)}(x_0, Re^{Kt}), \tag{2.1}$$

*or equivalently, for all $y \in B_{g(0)}(x_0, Re^{Kt})$ we have*

$$d_{g(t)}(y, x_0) \geq d_{g(0)}(y, x_0)e^{Kt}. \tag{2.2}$$

**Lemma 2.2.** *(Shrinking Lemma). Given $T, c_0, r > 0$ and $n \in \mathbb{N}$, there exists constant $\beta = \beta(n) \geq 1$ such that the following holds: Let $(M^n, g(t))$ be a Ricci flow for $t \in [0, T]$. Suppose for some $x_0 \in M$ we have $B_{g(0)}(x_0, r) \subset\subset M$. Suppose also $|Rm|_{g(t)} \leq \frac{c_0}{t}$, or more generally $Ric_{g(t)} \leq \frac{(n-1)c_0}{t}$, on $B_{g(0)}(x_0, r) \cap B_{g(t)}(x_0, r - \beta\sqrt{c_0 t})$ for each $t \in [0, T]$.*

*Then for all $t \in [0, T]$, we have*

$$B_{g(0)}(x_0, r) \supset B_{g(t)}(x_0, r - \beta\sqrt{c_0 t}), \tag{2.3}$$

*or equivalently, for all $y \in B_{g(t)}(x_0, r - \beta\sqrt{c_0 t})$ we have*

$$d_{g(t)}(y, x_0) \geq d_{g(0)}(y, x_0) - \beta\sqrt{c_0 t}. \tag{2.4}$$

*More generally, for $0 \leq s \leq t \leq T$, we have*

$$B_{g(s)}(x_0, r - \beta\sqrt{c_0 s}) \supset B_{g(t)}(x_0, r - \beta\sqrt{c_0 t}), \tag{2.5}$$

*or equivalently, for all $y \in B_{g(t)}(x_0, r - \beta\sqrt{c_0 t})$ we have*

$$d_{g(t)}(y, x_0) \geq d_{g(s)}(y, x_0) - \beta(\sqrt{c_0 t} - \sqrt{c_0 s}). \tag{2.6}$$

As an application of the Shrinking Lemma, we get the following Hölder estimate [24, Lemma 3.1].

**Lemma 2.3.** *Given $T, c_0, r > 0$ and $n \in \mathbb{N}$, there exist positive constants $\beta = \beta(n)$ and $\gamma = \gamma(c_0, n, T)$ such that the following holds: Let $(M^n, g(t))$ be a Ricci flow for $t \in [0, T]$, not necessarily complete. Suppose for some $x_0 \in M$, we have $B_{g(t)}(x_0, 2r) \subset\subset M$ for all $t \in [0, T]$. Suppose also $|Rm|_{g(t)}(x) \leq \frac{c_0}{t}$, or more generally $Ric_{g(t)}(x) \leq \frac{(n-1)c_0}{t}$ for all $x \in B_{g(t)}(x_0, 2r)$ and $t \in [0, T]$.*

*Then for all* $x, y \in \bigcap_{s\in[0,T]} B_{g(s)}(x_0, r)$, *and* $0 \le t_1 < t_2 \le T$, *we have*

$$d_{g(t_2)}(x, y) \ge d_{g(t_1)}(x, y) - \beta\sqrt{c_0}(\sqrt{t_2} - \sqrt{t_1}). \tag{2.7}$$

*Moreover, for all* $t \in [0, T]$, *we have*

$$d_{g(t)}(x, y) \ge \gamma[d_{g(0)}(x, y)]^{1+2(n-1)c_0}. \tag{2.8}$$

**Remark 2.4.** We need the curvature assumption on $B_{g(t)}(x_0, 2r) \subset\subset M$ for all $t$ to estimate the distances change on $\cap_{s\in[0,T]} B_{g(s)}(x_0, r)$. The reason is that there are two ways to make sense of the distance at time $t$ between two points $x, y \in B_{g(t)}(x_0, 2r)$. One is the infimum length of all connecting paths in $M$, and the other is the infimum length of all connecting paths that are contained in $B_{g(t)}(x_0, 2r)$. The former is usually shorter than the latter. These two metrics agree for $x, y \in B_{g(t)}(x_0, r)$ when $B_{g(t)}(x_0, 2r)$ is compactly contained in $M$, and the distance can be realized by a geodesic which lies within $B_{g(t)}(x_0, 2r)$.

**Remark 2.5.** We can also prove the same conclusion for the Ricci flow defined only for $t \in (0, T]$, where $d_{g(0)}$ in (2.8) is replaced by the limit distance of $d_{g(t)}$. The limit exists thanks to bound $|\mathrm{Rm}|_{g(t)} \le \frac{C}{t}$ in (1.2).

**Proof of Lemma 2.3.** We note that there is no ambiguity to talk about $d_{g(t)}(x, y)$ for $x, y \in \bigcap_{s\in(0,T]} B_{g(s)}(x_0, r)$ for all $t \in [0, T]$, because the minimizing geodesic joining $x$ and $y$ with respect to $g(t)$ is contained in $B_{g(t)}(x_0, 2r) \subset\subset M$. Inequality (2.7) follows by the above Shrinking Lemma. The proof of (2.8) follows by splitting $[0, t]$ into two intervals. We choose and fix $t_0 = \frac{1}{c_0}\left[\frac{1}{2\beta} d_{g(0)(x,y)}\right]^2$. Then in the first interval $[0, t_0]$, we integrate the following inequality from Hamilton and Perelman

$$\frac{\partial^+}{\partial t} d_{g(t)}(x, y) \ge -\frac{\beta}{2}\sqrt{\frac{c_0}{t}} \tag{2.9}$$

to get

$$d_{g(t_0)}(x, y) \ge \frac{1}{2} d_{g(0)}(x, y). \tag{2.10}$$

By $\frac{\partial^+}{\partial t}|_{t_0} F$ we mean $\limsup_{t\to t_0^+} \frac{F(t)-F(t_0)}{t-t_0}$. In the second interval we use the following inequality, which follows from the Ricci flow equation

$$\frac{\partial^+}{\partial t} d_{g(t)}(x, y) \ge -(n-1)\frac{c_0}{t} d_{g(t)}(x, y), \tag{2.11}$$

integrating which we get

$$d_{g(t)}(x,y) \geq d_{g(t_0)}(x,y)\left[\frac{t}{t_0}\right]^{-(n-1)c_0}. \tag{2.12}$$

The combination of (2.10) and (2.12) gives (2.8). □

### 2.2. Extension lemma

For the metric on a local region, we can modify it by a conformal change that pushes the boundary of the region, on which we have curvature bounds, to infinity in such a way that the modified metric is complete and has bounded curvature. For example, the open Euclidean unit ball can be made into a complete hyperbolic metric under a conformal change. The following conformal change has been used in [13], [24]. In [1], a different conformal change was also used to achieve the local results of the first and third cases listed in Theorem 1.1, as a corollary of their corresponding global results.

**Lemma 2.6.** *(Conformal Change Lemma) Let $(N^n, g)$ be a smooth (not necessarily complete) Riemannian manifold and let $U \subset N$ be an open set. Assume that for some $\rho \in (0,1]$, we have $\sup_U |Rm|_g \leq \rho^{-2}$, $B_g(x,\rho) \subset\subset N$ and $inj_g(x) \geq \rho$ for all $x \in U$. Then there exist a constant $\gamma = \gamma(n) \geq 1$, an open set $\tilde{U} \subset U$ and a smooth metric $\tilde{g}$ defined on $\tilde{U}$ such that each connected component of $(\tilde{U}, \tilde{g})$ is a complete Riemannian manifold satisfying*

1. *$|Rm|_{\tilde{g}} \leq \gamma\rho^{-2}$ and $inj_{\tilde{g}} \geq \frac{1}{\sqrt{\gamma}}\rho$ for $x \in \tilde{U}$*
2. *$U_\rho \subset \tilde{U} \subset U$*
3. *$\tilde{g} = g$ on $\tilde{U}_\rho \supset U_{2\rho}$,*

*where $U_s = \{x \in U | B_g(x,s) \subset\subset U\}$.*

### 2.3. Some integrations

For later convenience, we include some frequently used inequalities and their proofs in this subsection.

**Lemma 2.7.** *Given $K, R, C_1 > 0$, $t \in (0,1]$ and $n \in \mathbb{N}$. There exists positive constant $C = C(K, C_1, n)$ such that the following holds. Let $(M, g)$ be a complete Riemannian manifold with $Ric \geq -(n-1)K$ on $B_g(x,R)$ for some point $x \in M$. Then*

$$\frac{C_1}{t^{\frac{n}{2}}} \int\limits_{B_g(x,R)} exp\left(-\frac{d_g^2(x,y)}{C_1 t}\right) d_g y \leq C \tag{2.13}$$

**Proof.** Let $\hat{g} = \frac{1}{t}g$, then it suffices to show $I := C_1 \int_{B_{\hat{g}}(x,\frac{R}{\sqrt{t}})} exp(-\frac{d_{\hat{g}}^2(x,y)}{C_1}) d_{\hat{g}} y \leq C(C_1, K, n)$. For all $y \in B_{\hat{g}}(x, \frac{R}{\sqrt{t}})$, the minimizing geodesic connecting $x$ and $y$ lies

within $B_{\hat{g}}(x, \frac{R}{\sqrt{t}})$ where Ric $\geq -(n-1)Kt \geq -(n-1)K$. So by Laplacian comparison the volume form $d_{\hat{g}}y \leq sn_{-K}^{n-1}(r(y))dr \wedge dvol_{n-1} \leq \frac{exp((n-1)\sqrt{K}r)}{(2\sqrt{K})^{n-1}} dr \wedge dvol_{n-1}$, where $r$ is the distance function centered at $x$ and $dvol_{n-1}$ is the standard volume form on $S^{n-1}(1)$. So we can express the integral on the segment domain in $T_xM$ and obtain

$$\begin{aligned} I &\leq \frac{C_1}{(2\sqrt{K})^{n-1}} \int\limits_{r \leq \frac{R}{\sqrt{t}}} exp\left(-\frac{r^2}{C_1}\right) exp((n-1)\sqrt{K}r)\, dr \wedge dvol_{n-1} \\ &\leq C(C_1, n, K) \int\limits_{\mathbb{R}} exp\left(-\frac{r^2}{C_1} + (n-1)\sqrt{K}r\right) dr \leq C(C_1, n, K) \quad \square \end{aligned}$$

**Lemma 2.8.** *Given $C_1, C_2 > 0$ and $n \in \mathbb{N}$. Let $(M, g(t)), t \in [0,1]$ be a complete Ricci flow with $|Rm|_{g(t)} \leq \frac{C_1}{t}$. Then for any $d \geq 2(n-1)\sqrt{C_1}C_2$,*

$$\frac{C_2}{t^{\frac{n}{2}}} \int\limits_{M - B_{g(t)}(x, \sqrt[4]{t}\, d)} exp\left(-\frac{d_t^2(x,y)}{C_2 t}\right) d_t y \leq C\, exp\left(-\frac{d^2}{C\sqrt{t}}\right) \tag{2.14}$$

*where $C$ is a constant depending on $n$, $C_1$ and $C_2$.*

**Proof.** For convenience, $C$ denotes all the constants depending on $C_1$, $C_2$, and $C_3$. Fix $t$, let $\hat{g} = \frac{1}{t} g(t)$. Then it suffices to show

$$C_2 \int\limits_{M - B_{\hat{g}}(x, \frac{d}{\sqrt[4]{t}})} exp\left(-\frac{d_{\hat{g}}^2(x,y)}{C_2}\right) d_{\hat{g}} y \leq C\, exp\left(-\frac{d^2}{C\sqrt{t}}\right) \tag{2.15}$$

with $|\mathrm{Rm}|_{\hat{g}} \leq C_1$.

Since Ric $\geq -(n-1)C_1$, we get by Laplacian comparison that the volume form $d_{\hat{g}}y \leq sn_{-C_1}^{n-1}(r(y))dr \wedge dvol_{n-1} \leq \frac{e^{(n-1)\sqrt{C_1}r}}{(2\sqrt{C_1})^{n-1}} dr \wedge dvol_{n-1}$ Thus by considering the integral over the segment domain in $T_xM$, denoting by $\omega_{n-1}$ the volume of $S^{n-1}(1)$, we get

$$\begin{aligned} I &\leq \frac{C_2}{(2\sqrt{C_1})^{n-1}} \int\limits_{r \geq \frac{d}{\sqrt[4]{t}}} exp\left(-\frac{r^2}{C_2}\right) exp((n-1)\sqrt{C_1}r)\, dr \wedge dvol_{n-1} \\ &= \frac{C_2}{(2\sqrt{C_1})^{n-1}} \omega_{n-1} \int\limits_{r \geq \frac{d}{\sqrt[4]{t}}} exp\left(-\frac{r^2}{C_2} + (n-1)\sqrt{C_1}r\right) dr \\ &\leq C \int\limits_{r \geq \frac{d}{\sqrt[4]{t}}} exp\left(-\frac{r^2}{2C_2}\right) dr \leq C\, exp\left(-\frac{d^2}{2C_2\sqrt{t}}\right). \quad \square \end{aligned}$$

**Lemma 2.9.** *Given $t, T, d, C > 0$ and $n \in \mathbb{N}$ such that $t < T \leq d^2$, there exists positive constant $C_1 = C_1(C, n)$ such that*

$$\frac{C}{t^{\frac{n}{2}}} exp\left(-\frac{d^2}{Ct}\right) \leq \frac{C_1}{T^{\frac{n}{2}}} exp\left(-\frac{d^2}{C_1 T}\right). \tag{2.16}$$

**Proof.** It's easy to see there exists $C_1 = C_1(C, n)$ such that for all $x > 0$,

$$\frac{1}{x^{\frac{n}{2}}} exp\left(-\frac{1}{Cx}\right) \leq C_1 exp\left(-\frac{1}{2Cx}\right). \tag{2.17}$$

Then (2.16) follows immediately from this inequality and the above assumptions. □

### *2.4. Weak derivatives*

In this section, we assume $(M^n, g(t))$ is a complete Ricci flow with bounded curvature. As we mentioned in introduction, $\ell$ satisfies the evolution inequality (1.7) in the barrier sense: for any $(q, \tau) \in M \times (0, T)$ we find a neighborhood $\mathcal{U} \subset M \times (0, T)$ of $(q, \tau)$ and a $C^\infty$ function $\phi : \mathcal{U} \to \mathbb{R}$ such that $\phi \leq \ell$ on $\mathcal{U}$, with equality at $(q, \tau)$ and

$$(\frac{\partial}{\partial t} - \Delta)\phi \leq \text{scal}\ell + C(n)\ell^2 \quad \text{at} \quad (q, \tau). \tag{2.18}$$

Set $\mathcal{L} = e^{-C(n)t}\ell$ and assume $\ell \leq 1$ then by (1.7) we have the following inequality which holds in the barrier sense

$$(\frac{\partial}{\partial t} - \Delta)\mathcal{L} \leq \text{scal}\,\mathcal{L}. \tag{2.19}$$

Suppose for a moment that $\mathcal{L}$ is smooth and $\psi(x, t)$ is a non-negative smooth function which is compactly supported in $M$ for each $t$. Then we see from the integration by parts formula that

$$\begin{aligned}
\frac{\partial}{\partial t} \int_U \mathcal{L}\psi \, d_t x &= \int_U (\frac{\partial}{\partial t}\mathcal{L}\psi - \mathcal{L}\psi\,\text{scal} + \mathcal{L}\frac{\partial}{\partial t}\psi) \, d_t x \\
&\leq \int_U ((\Delta\mathcal{L})\psi + \mathcal{L}\frac{\partial}{\partial t}\psi) \, d_t x \\
&= \int_U \mathcal{L}(\Delta\psi + \frac{\partial}{\partial t}\psi) \, d_t x.
\end{aligned} \tag{2.20}$$

We show in Lemma 2.11 that some variant of (2.20) is still true without the smooth assumptions either on $\ell$ or the test function $\psi$.

First, we give the definitions of inequalities in several weak senses. We say a continuous function $f : M \to \mathbb{R}$ satisfies $\Delta f \leq u$ for some function $u : M \to \mathbb{R}$ in the barrier sense if

for any point $x$ and every $\varepsilon > 0$ there exists a neighborhood $\mathcal{U}_\varepsilon \subset M$ of $x$ and a smooth function $h_\varepsilon : \mathcal{U}_\varepsilon \to \mathbb{R}$ such that $h_\varepsilon(x) = f(x)$, $h_\varepsilon \geq f$ in $\mathcal{U}_\varepsilon$ and $\Delta h_\varepsilon(x) \leq u(x) + \varepsilon$.

We say a continuous function $f : M \to \mathbb{R}$ satisfies $\Delta f \leq u$ for some bounded function $u : M \to \mathbb{R}$ in the distributional sense if for any non-negative smooth function $h$ with compact support one has $\int f \Delta h \leq \int uh$. By standard argument, if $f$ satisfies $\Delta f \leq u$ in the barrier sense, then $f$ satisfies it in the distributional sense (see for example [14, Appendix A]).

**Lemma 2.10.** *Let $\psi(x,t)$ be a non-negative smooth function which is compactly supported in $M$ for each $t$. $\mathcal{L} = e^{-C(n)t}\ell$ with $\ell \leq 1$. Then we have*

$$\frac{\partial^+}{\partial t} \int \mathcal{L}\psi \, d_t x \leq \int \mathcal{L}(\Delta\psi + \frac{\partial}{\partial t}\psi) \, d_t x \tag{2.21}$$

*for all $t \in [a,b) \subset (0,T)$, integrating which we have:*

$$(\int \mathcal{L}\psi \, d_t x)\Big|_a^b \leq \int_a^b (\int \mathcal{L}(\Delta\psi + \frac{\partial}{\partial t}\psi)) \, d_t x \, dt \tag{2.22}$$

**Proof.** Let $t_0$ be an arbitrary time in $[a,b)$. Since $\mathcal{L}$ satisfies

$$(\frac{\partial}{\partial t} - \Delta)\mathcal{L} \leq \operatorname{scal} \mathcal{L}$$

in the barrier sense, by the maximum principle for complete manifold with bounded curvature, $\mathcal{L}(\cdot,t) \leq \overline{\mathcal{L}}(\cdot,t)$ for all $t \in [t_0,b]$, where $\overline{\mathcal{L}}$ is the solution to the initial value problem:

$$(\frac{\partial}{\partial s} - \Delta)\overline{\mathcal{L}} = \operatorname{scal}\overline{\mathcal{L}}, \quad \overline{\mathcal{L}}(\cdot,t_0) = \mathcal{L}(\cdot,t_0). \tag{2.23}$$

Then $\overline{\mathcal{L}}$ is smooth for all $t > t_0$ and so we have

$$\frac{\partial^+}{\partial t}\Big|_{t_0} \int \mathcal{L}\psi \, d_t x \leq \frac{\partial^+}{\partial t}\Big|_{t_0} \int \overline{\mathcal{L}}\psi \, d_t x = \lim_{t \to t_0^+} \frac{\partial}{\partial t} \int \overline{\mathcal{L}}\psi \, d_t x. \tag{2.24}$$

For each $t > t_0$, we calculate by integration by parts to get

$$\frac{\partial}{\partial t} \int \overline{\mathcal{L}}\psi \, d_t x = \int \overline{\mathcal{L}}(\Delta\psi + \frac{\partial}{\partial t}\psi) \, d_t x, \tag{2.25}$$

substituting which into (2.24) we have

$$\frac{\partial^+}{\partial t}\Big|_{t_0} \int \mathcal{L}\psi \, d_t x \leq \lim_{t \to t_0^+} \int \overline{\mathcal{L}}(\Delta\psi + \frac{\partial}{\partial t}\psi) \, d_t x$$

$$= \int \mathcal{L}(\Delta\psi + \frac{\partial}{\partial t}\psi)\, d_t x\Big|_{t_0} \qquad \square \tag{2.26}$$

**Lemma 2.11.** *Let $\psi(x,t)$ be a non-negative continuous function which is compactly supported in $M$ for each $t$, and satisfies $\Delta\psi \le u(x,t)$ and $\frac{\partial}{\partial t}\psi \le v(x,t)$ in the barrier sense, where $v(x,t)$ is continuous with respect to $t$.*

*Then for all $t$ we have*

$$\frac{\partial^+}{\partial t}\int \mathcal{L}(x,t)\psi(x,t)\, d_t x \le \int \mathcal{L}(x,t)(u(x,t)+v(x,t))\, d_t x \tag{2.27}$$

**Proof.** Let $t_0$ be an arbitrary time in $(a,b)$. Differentiating at $t_0$ by the product rule we get

$$\begin{aligned}&\frac{\partial^+}{\partial t}\Big|_{t_0}\int \mathcal{L}(x,t)\psi(x,t)\, d_t x\\ &\quad\le \int \mathcal{L}(x,t_0)v(x,t_0)\, d_{t_0}x + \frac{\partial^+}{\partial t}\Big|_{t_0}\int \mathcal{L}(x,t)\psi(x,t_0)\, d_t x.\end{aligned} \tag{2.28}$$

Let $\overline{\mathcal{L}}$ be the solution to the initial value problem

$$(\frac{\partial}{\partial s} - \Delta)\overline{\mathcal{L}} = \mathrm{scal}\overline{\mathcal{L}}, \quad \overline{\mathcal{L}}(\cdot,t_0) = \mathcal{L}(\cdot,t_0). \tag{2.29}$$

Then $\overline{\mathcal{L}}$ is smooth for all $t > t_0$. We calculate using the fact that barrier sense implies distributional sense:

$$\begin{aligned}\frac{\partial^+}{\partial t}\Big|_{t_0}\int \mathcal{L}(x,t)\psi(x,t_0)\, d_t x &\le \frac{\partial^+}{\partial t}\Big|_{t_0}\int \overline{\mathcal{L}}(x,t)\psi(x,t)\, d_t x\\ &\le \limsup_{t\to t_0^+}\frac{\partial}{\partial t}\int \overline{\mathcal{L}}(x,t)\psi(x,t_0)d_t x\\ &= \limsup_{t\to t_0^+}\int \Delta\overline{\mathcal{L}}(x,t)\psi(x,t_0)d_t x\\ &\le \limsup_{t\to t_0^+}\int \overline{\mathcal{L}}(x,t)u(x,t_0)d_t x\\ &= \int \mathcal{L}(x,t_0)u(x,t_0)d_{t_0}x\end{aligned} \tag{2.30}$$

where we used the fact that barrier sense implies distributional sense. $\square$

## 3. Curvature decay lemma

The main result in this section is Lemma 3.4, which provides a local estimate on the norm of the Riemann curvature tensor, under the assumption of a local bound for $\ell$.

This lemma can be viewed as a weaker version of Theorem 1.1 in the sense that we take the two conclusions of the existence of the Ricci flow and the bound of $\ell$, as additional hypotheses, and deduce the remaining conclusion about $|\mathrm{Rm}|$.

We need three ingredients in the proof of Lemma 3.4. One is the following Lemma, given in [23, Lemma 5.1] by a point-picking argument.

**Lemma 3.1.** *Given $c_0, r_0 > 0$, $n \in \mathbb{N}$, and take $\beta = \beta(n) > 0$ as in Lemma 2.2. Let $(M^n, g(t))$, $t \in [0,T]$ be a Ricci flow. Suppose for some $x_0 \in M$ we have $B_{g(t)}(x_0, r_0) \subset\subset M$ for each $t \in [0,T]$.*

*Then at least one of the following assertions is true:*

1. *For each $t \in [0,T]$ with $t < \frac{r_0^2}{\beta^2 c_0}$, we have $B_{g(t)}(x_0, r_0 - \beta\sqrt{c_0 t}) \subset B_{g(0)}(x_0, r_0)$ and*

$$|Rm|_{g(t)} < \frac{c_0}{t} \quad \text{on } B_{g(t)}(x_0, r_0 - \beta\sqrt{c_0 t}). \tag{3.1}$$

2. *There exist $\bar{t} \in (0,T]$ with $\bar{t} < \frac{r_0^2}{\beta^2 c_0}$ and $\bar{x} \in B_{g(\bar{t})}(x_0, r_0 - \frac{1}{2}\beta\sqrt{c_0 \bar{t}})$ such that*

$$Q := |Rm|_{g(\bar{t})}(\bar{x}) \geq \frac{c_0}{\bar{t}}, \tag{3.2}$$

*and*

$$|Rm|_{g(t)}(x) \leq 4Q = 4|Rm|_{g(\bar{t})}(\bar{x}), \tag{3.3}$$

*whenever $d_{g(\bar{t})}(x, \bar{x}) < \frac{\beta c_0}{8} Q^{-\frac{1}{2}}$ and $\bar{t} - \frac{1}{8} c_0 Q^{-1} \leq t \leq \bar{t}$.*

The second ingredient we need, [23, Lemma 2.3], says that the volume of a ball of fixed radius cannot decrease too rapidly under some curvature hypothesis.

**Lemma 3.2.** *Given $K, \gamma, c_0, v_0, T > 0$ and $n \in \mathbb{N}$, there exist positive constants $\varepsilon_0 = \varepsilon_0(v_0, K, \gamma, n)$ and $\hat{T} = \hat{T}(v_0, c_0, K, \gamma, n) \geq 0$ such that the following holds: Let $(M^n, g(t)), t \in [0,T)$ be a Ricci flow such that $B_{g(t)}(x_0, \gamma) \subset\subset M$ for some $x_0 \in M$ and all $t \in [0,T)$. Suppose $Ric_{g(t)} \geq -K$ and $|Rm|_{g(t)} \leq \frac{c_0}{t}$ on $B_{g(t)}(x_0, \gamma)$ for all $t \in [0,T)$, and $Vol_{g(0)} B_{g(0)}(x_0, \gamma) \geq v_0$.*

*Then*

$$Vol_{g(t)} B_{g(t)}(x_0, \gamma) \geq \varepsilon_0 \tag{3.4}$$

*for all $t \in [0, \hat{T}] \cap [0,T)$.*

The third ingredient is the following Lemma, which says that the asymptotic volume ratio of a weakly $\mathrm{PIC}_1$ ancient solution is zero. This is proved in [1, Lemma 4.2]. We note that each curvature condition listed in Theorem 1.1 implies weakly $\mathrm{PIC}_1$, so the proof of Lemma 3.4 is uniform for all $\mathcal{C}$.

**Lemma 3.3.** *Let $(M^n, g(t)), t \in (-\infty, 0]$ be a nonflat ancient solution of the Ricci flow with bounded curvature satisfying weakly PIC$_1$. Then it has non-negative complex sectional curvature. Furthermore, the volume growth is non-Euclidean, i.e. $\lim_{r\to\infty} r^{-n} Vol_{g(0)} B_{g(0)}(x, r) = 0$ for all $x \in M$.*

We now state our main result of this section. In the proof we blow up a contradicting sequence to get a weakly PIC$_1$ ancient solution with positive asymptotic volume ratio, which is impossible by Lemma 3.3.

**Lemma 3.4.** *(Curvature Decay Lemma). Given $v_0, K > 0$, there exist positive constants $\tilde{T} = \tilde{T}(v_0, K, n)$, $C_1 = C_1(v_0, K, n)$ and $\eta_0 = \eta_0(v_0, K, n)$ such that the following holds: Let $(M^n, g(t)), t \in [0, T]$ be a Ricci flow (not necessarily complete) such that $B_{g(t)}(x_0, 1) \subset\subset M$ for each $t \in [0, T]$ and some $x_0 \in M$, and*

$$Vol_{g(0)} B_{g(0)}(x_0, 1) \geq v_0 > 0. \tag{3.5}$$

*Suppose further that*

$$\ell(x, t) \leq K \quad \text{on} \quad \bigcup_{s\in[0,T]} B_{g(s)}(x_0, 1), \quad \text{for all} \ \ t \in [0, T]. \tag{3.6}$$

*Then for all $t \in (0, T) \cap (0, \tilde{T})$, we have*

$$|Rm|_{g(t)} < \frac{C_1}{t} \quad \text{on} \quad B_{g(t)}(x_0, \frac{1}{2}), \tag{3.7}$$

*and*

$$Vol_{g(t)} B_{g(t)}(x_0, 1) \geq \eta_0 \quad \text{and} \quad inj_{g(t)}(x_0) \geq \sqrt{\frac{t}{C_1}} \tag{3.8}$$

*for all $t \in (0, \min(T, \tilde{T})]$.*

**Proof.** By Bishop–Gromov inequality, $Vol_{g(0)} B_{g(0)}(x_0, \frac{1}{2})$ has a positive lower bound depending only on $v_0$, $n$ and $K$. Applying Lemma 3.2 to $g(t)$, we see that there exists $\eta_0 > 0$ depending only on $v_0$, $n$ and $K$ such that for each $C_1 < \infty$, there exist $\tilde{T} = \tilde{T}(v_0, n, C_1)$ such that prior to time $\tilde{\mathrm{T}}$ and while $|\mathrm{Rm}|_{g(t)} \leq \frac{C_1}{t}$ still holds on $B_{g(t)}(x_0, \frac{1}{2})$, we have a lower volume bound

$$Vol_{g(t)} B_{g(t)}(x_0, 1) \geq \eta_0. \tag{3.9}$$

In particular, $\eta_0$ is independent of $C_1$. From this we deduce that is suffices to prove the lemma with the additional hypothesis that the equation above holds for each $t \in [0, T)$.

Let us assume that the lemma is false, even with the extra hypothesis. Then for any sequence $c_k \to \infty$, we can find Ricci flows that fail the lemma with $C_1 = c_k$ in an

arbitrary short time, and in particular within a time $t_k$ that is sufficiently small so that $c_k t_k \to 0$ as $k \to \infty$. By reducing $t_k$ to the first time at which the desired conclusion fails, we have a sequence of Ricci flows $(M_k, \tilde{g}_k(t))$ for $t \in [0, t_k]$ with $t_k \to 0$, and even $c_k t_k \to 0$, and a sequence of points $x_k \in M_k$ with $B_{\tilde{g}_k(t)}(x_k, 1) \subset\subset M_k$ for each $t \in [0, t_k]$, such that

$$Vol_{\tilde{g}_k(t)} B_{\tilde{g}_k(t)}(x_k, 1) \geq \eta_0, \text{ for all } t \in [0, t_k], \tag{3.10}$$

$$\ell(x,t) \leq K, \text{ on } \bigcup_{s \in [0,t_k]} B_{\tilde{g}_k(s)}(x_k, 1) \text{ for all } t \in [0, t_k], \tag{3.11}$$

and

$$|\mathrm{Rm}|_{\tilde{g}_k(t)} < \frac{c_k}{t} \text{ on } B_{\tilde{g}_k(t)}(x_k, \frac{1}{2}) \text{ for all } t \in [0, t_k], \tag{3.12}$$

but so that

$$|\mathrm{Rm}|_{\tilde{g}_k(t_k)} = \frac{c_k}{t_k} \text{ at some point in } \overline{B_{\tilde{g}_k(t)}(x_k, \frac{1}{2})}. \tag{3.13}$$

For sufficiently large $k$, we have $\beta\sqrt{c_k t_k} < \frac{1}{4}$. We apply Lemma 3.1, to each $\tilde{g}_k(t)$ with $r_0 = \frac{3}{4}$ and $c_0 = c_k$, then it follows by (3.13) that Assertion 1 there cannot hold, and thus Assertion 2 must hold for each $k$, giving time $\bar{t}_k \in (0, t_k]$ and points $\bar{x}_k \in B_{\tilde{g}_k(\bar{t}_k)}(x_k, r_0 - \frac{1}{2}\beta\sqrt{c_k \bar{t}_k})$ such that

$$|\mathrm{Rm}|_{\tilde{g}_k(t)}(x) \leq 4|\mathrm{Rm}|_{\tilde{g}_k(\bar{t}_k)}(\bar{x}_k) \tag{3.14}$$

on $B_{\tilde{g}(\bar{t}_k)}(\bar{x}_k, \frac{\beta c_k}{8} Q_k^{-\frac{1}{2}})$, for all $t \in [\bar{t}_k - \frac{1}{8} c_k Q_k^{-1}, \bar{t}_k]$, where $Q_k := |\mathrm{Rm}|_{\tilde{g}_k(\bar{t}_k)}(\bar{x}_k) \geq \frac{c_k}{t_k} \to \infty$. We also notice that $B_{\tilde{g}(\bar{t}_k)}(\bar{x}_k, \frac{\beta c_k}{8} Q_k^{-\frac{1}{2}}) \subset B_{\tilde{g}(\bar{t}_k)}(x_k, 1)$, thus

$$\ell(x,t) \leq K \tag{3.15}$$

on $B_{\tilde{g}(\bar{t}_k)}(\bar{x}_k, \frac{\beta c_k}{8} Q_k^{-\frac{1}{2}}) \times [\bar{t}_k - \frac{1}{8} c_k Q_k^{-1}, \bar{t}_k]$. The above conditions at $\bar{t}_k$, together with Bishop–Gromov inequality, imply that we have uniform volume ratio control

$$\frac{Vol_{\tilde{g}_k(\bar{t}_k)} B_{\tilde{g}_k(\bar{t}_k)}(\bar{x}_k, r)}{r^n} \geq \eta > 0 \tag{3.16}$$

for all $0 < r < \frac{1}{4}$, where $\eta$ depends on $\eta_0$, $K$ and $n$. A parabolic rescaling on $B_{\tilde{g}(\bar{t}_k)}(\bar{x}_k, \frac{\beta c_k}{8} Q_k^{-\frac{1}{2}}) \times [\bar{t}_k - \frac{1}{8} c_k Q_k^{-1}, \bar{t}_k]$ gives new Ricci flows defined by

$$g_k(t) := Q_k \tilde{g}_k(\frac{t}{Q_k} + \bar{t}_k)$$

for $t \in [-\frac{1}{8}c_k, 0]$. The scaling factor is chosen so that $|\mathrm{Rm}|_{g_k(0)}(\bar{x}_k) = 1$. By (3.14), the curvature of $g_k(t)$ is uniformly bounded on $B_{g_k(0)}(\bar{x}_k, \frac{1}{8}\beta c_k) \times [-\frac{1}{8}c_k, 0]$. Condition (3.15) transforms to

$$\ell(x,t) \leq \frac{K}{Q_k} \to 0 \tag{3.17}$$

on $B_{g_k(0)}(\bar{x}_k, \frac{1}{8}\beta c_k) \times [-\frac{1}{8}c_k, 0]$. The volume ratio (3.16) gives

$$\frac{Vol_{g_k(0)} B_{g_k(0)}(\bar{x}_k, r)}{r^n} \geq \eta > 0 \tag{3.18}$$

for all $0 < r < \frac{1}{4}Q_k^{\frac{1}{2}} \to \infty$.

With this control we can apply Hamilton's compactness theorem to give convergence $(M_k, g_k(t), \bar{x}_k) \to (N, g(t), x_\infty)$, for some complete bounded-curvature Ricci flow $(N, g(t))$, for $t \in (-\infty, 0]$, and $x_\infty \in N$.

Moreover, the last volume equation passes to limit to force $g(t)$ to have positive asymptotic volume ratio. From (3.17) we know that $g(t)$ is a nonflat ancient solution of Ricci flow with bounded curvature satisfying weakly $\mathrm{PIC}_1$. This contradicts Lemma 3.3 that the volume ratio of $(N, g(t))$ vanishes, and thus shows the first part of the Lemma. For the second part, the injectivity radius estimate of Cheeger–Gromov–Taylor [4] and the Bishop–Gromov comparison then tell us $\mathrm{inj}_{g(t)}(x) \geq i_0\sqrt{t}$ for some $i_0 = i_0(\eta_0, C) > 0$. □

## 4. A cut-off function

In this section we construct a cut-off function on manifolds (not assumed to be complete) evolving by Ricci flow, which helps to localize the integration estimates in section 7.

**Lemma 4.1.** *Given $n \in \mathbb{N}$, $c_0, K > 0$, $0 < T < 1$, $0 < R < 1$, $0 < r < \frac{1}{10}$ with $\beta\sqrt{c_0 T} \leq \frac{1}{4}r$, where $\beta = \beta(n)$ is from the Shrinking Lemma, there exists positive constant $C = C(n, K, v_0)$ such that the following holds: Let $(M^n, g(t)), t \in [0, T]$ be a smooth Ricci flow such that $B_{g(0)}(x_0, R + r) \subset\subset M$, and on $B_{g(0)}(x_0, R + r) \times [0, T]$,*

$$Ric_{g(t)}(x) \geq -K \quad \text{and} \quad |Rm|_{g(t)} \leq \frac{c_0}{t}, \tag{4.1}$$

*and for all $\delta \in [0, r]$ and $x \in B_{g(0)}(x_0, R)$ we have*

$$Vol_{g(0)} B_{g(0)}(x, \delta) \geq v_0 \delta^n. \tag{4.2}$$

*Then there exists a continuous function $\phi(y, s) : M \times [0, T] \longrightarrow \mathbb{R}$ with the following properties:*

**(P1)** *supp* $\phi(\cdot, s) \subset B_{g(0)}(x_0, R)$ *for all* $s \in [0, T]$.
**(P2)** $\nabla\phi$ *exists a.e. and* $|\nabla\phi| \leq Cr^{-(n+1)}$.
**(P3)** $\Delta\phi \leq Cr^{-(2n+2)}$ *in the barrier sense.*
**(P4)** $\frac{\partial^+}{\partial s}\phi \leq Cr^{-n}$.

*Moreover, we have the inclusions:*

$$B_{g(s)}(x_0, R - \tfrac{5}{4}r) \subset B_{g(0)}(x_0, R - r) \subset \{y \in M \mid \phi(y, s) = 1\} \tag{4.3}$$

*for all* $s \in [0, T]$.

**Proof.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-increasing smooth function such that $f(z) = 1$ for all $z < \frac{1}{4}$ and $f(z) = 0$ for all $z > \frac{1}{2}$. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-decreasing and convex smooth function such that $F(z) = 0$ for all $z \leq 0$ and $F(1) = 1$. Let $C_0$ be a constant such that $|f'|, |f''|, |F'|, |F''| \leq C_0$. Hereafter we use the same letter $C$ to denote the constants depending on $K$, $v_0$, $n$.

Let $\{p_k\}_{k=1}^N$ be a maximal $\frac{r}{4e^K}$-separated set in the annulus $A := B_{g(0)}(x_0, R) - B_{g(0)}(x_0, R - \frac{1}{4}r)$ with respect to $g(0)$. By a $\varepsilon$-separated set we mean a set in which the points are at least $\varepsilon$-distant from each other. It's clear that the $\varepsilon/2$-balls of points in a $\varepsilon$-separated set are disjoint pairwise. By volume comparison we see that $Vol_{g(0)}B_{g(0)}(x_0, R) \leq C$, and furthermore by (4.2) $Vol_{g(0)}B_{g(0)}(p_k, \frac{r}{4e^K}) \geq Cr^n$. Hence we have $N \leq Cr^{-n}$.

**Claim 4.2.** $A \subset \bigcup_{k=1}^{N} B_{g(s)}(p_k, \frac{r}{4})$ *for all* $s \in [0, T]$.

**Proof of Claim 4.2.** By the choice of $\{p_k\}_{k=1}^N$ we see that $A \subset \bigcup_{k=1}^{N} B_{g(0)}(p_k, \frac{r}{4e^K})$. For each $p_k$, the triangle inequality implies that $B_{g(0)}(p_k, \frac{r}{2}) \subset\subset B_{g(0)}(x_0, R + r)$ where $|\mathrm{Rm}|_{g(s)} \leq \frac{c_0}{s}$ and $\mathrm{Ric}_{g(s)} \geq -K$ holds true for all $s \in [0, T]$. Applying the Shrinking Lemma to $g(t)$, we find that $B_{g(s)}(p_k, \frac{r}{2} - \beta\sqrt{c_0 s}) \subset B_{g(0)}(p_k, \frac{r}{2})$ for all $s \in [0, T]$ and in particular $B_{g(s)}(p_k, \frac{r}{4}) \subset B_{g(0)}(p_k, \frac{r}{2})$ due to $\beta\sqrt{c_0 T} \leq \frac{1}{4}r$. So $\mathrm{Ric} \geq -K$ holds on $B_{g(s)}(p_k, \frac{r}{4})$, which gives the condition we need in order to apply the Expanding Lemma to the Ricci flow on $B_{g(s)}(p_k, \frac{r}{4}) \times [0, s]$, giving $B_{g(s)}(p_k, \frac{r}{4}) \supset B_{g(0)}(p_k, \frac{r}{4e^K})$, and thus proves the claim. □

By the Shrinking Lemma and triangle inequality, we have $B_{g(s)}(p_k, \frac{r}{2}) \subset B_{g(0)}(p_k, r) \subset B_{g(0)}(x_0, R + r)$. In view of this together with the definition of $f$, we define the following continuous function on $M$:

$$f_k(y, s) = \begin{cases} f\left(\frac{d_{g(s)}(p_k, y)}{r}\right) & \text{for} \quad y \in B_{g(0)}(p_k, r)\,; \\ 0 & \text{for} \quad y \notin B_{g(0)}(p_k, r). \end{cases} \tag{4.4}$$

By Claim 4.2, for each point $y \in A$ and $s \in [0,T]$, there is some $k$ such that $y \in B_{g(s)}(p_k, \frac{r}{4})$, $f_k(y,s) = 1$ and $F(1 - \sum_{k=1}^N f_k(y,s)) = 0$. Based on this we define the following continuous function on $M$:

$$\phi(y,s) = \begin{cases} F(1 - \sum\limits_{k=1}^N f_k(y,s)) & \text{for} \quad y \in B_{g(0)}(x_0, R)\ ; \\ 0 & \text{for} \quad y \notin B_{g(0)}(x_0, R)\ . \end{cases} \tag{4.5}$$

It's clear that $\phi(y,s)$ satisfies (P1). Below we abbreviate $d_{g(s)}(p_k, y)$ by $d_k$, $f'(\frac{d_{g(s)}(p_k,y)}{r})$ by $f'_k$, and $f''(\frac{d_{g(s)}(p_k,y)}{r})$ by $f''_k$. Using that

$$\nabla\phi = -F' \cdot \sum_{k=1}^N f'_k \cdot r^{-1} \cdot \nabla d_k, \tag{4.6}$$

and taking into account that $\nabla d_k$ exists a.e. with $|\nabla d_k| = 1$, and $N \le C \cdot r^{-n}$, we see that $\nabla\phi$ exists a.e. and

$$|\nabla\phi| \le C \cdot r^{-(n+1)}. \tag{4.7}$$

To estimate $\frac{\partial}{\partial s}\phi(y,s)$ and $\Delta\phi(y,s)$, we may assume $y \in B_{g(s)}(p_k, \frac{1}{2}r) - B_{g(s)}(p_k, \frac{1}{4}r)$ without loss of generality. Because otherwise $f'(\frac{d_k(y,s)}{r}) = 0$, and hence $\frac{\partial}{\partial s}\phi(y,s) = \Delta\phi(y,s) = 0$. By the Shrinking Lemma and the choice of $p_k$ we have

$$B_{g(s)}(p_k, \frac{1}{2}r) \subset B_{g(0)}(p_k, r) \subset B_{g(0)}(x_0, R + r). \tag{4.8}$$

So the minimizing geodesic connecting $y$ and $p_k$ with respect to $g(s)$ remains within $B_{g(0)}(x_0, R + r)$ where $\mathrm{Ric}_{g(s)} \ge -K$. Hence by the Laplacian comparison and noting that $d_{g(s)}(y, p_k) \ge \frac{1}{4}r$, we have

$$\Delta d_{g(s)}(p_k, y) \le (n-1)\sqrt{K}\coth(\sqrt{K}d_{g(s)}(p_k, y)) \le \frac{C}{r} \tag{4.9}$$

in the barrier sense. Then using that

$$\Delta\phi = F''|\sum_{k=1}^N f'_k \cdot r^{-1} \cdot \nabla d_k|^2 - F' \cdot \sum_{k=1}^N (f''_k \cdot r^{-2} \cdot |\nabla d_k|^2 + f'_k \cdot r^{-1} \cdot \Delta d_k), \tag{4.10}$$

and noting $f' \le 0$, $F' \ge 0$, we can estimate

$$\Delta\phi \le C \cdot r^{-(2n+2)}. \tag{4.11}$$

We see from the Ricci flow equation that

$$\frac{\partial^+}{\partial s} d_{g(s)}(p_k, y) \leq K d_{g(s)}(p_k, y) \leq \frac{1}{2} K r, \tag{4.12}$$

and using that

$$\frac{\partial^+}{\partial s} \phi = -F' \sum_{k=1}^{N} f_k' \cdot r^{-1} \cdot \frac{\partial^+}{\partial s} d_k, \tag{4.13}$$

we obtain

$$\frac{\partial^+}{\partial s} \phi \leq C \cdot r^{-n}. \tag{4.14}$$

It remains to prove the inclusion (4.3). The first inclusion is a consequence of the Shrinking Lemma and $\beta\sqrt{c_0 T} \leq \frac{1}{4} r$. To prove the second inclusion, we note by triangle inequality that

$$B_{g(0)}(x_0, R - r) \cap \bigcup_{k=1}^{N} B_{g(0)}(p_k, \frac{3}{4} r) = \emptyset, \tag{4.15}$$

and by the Shrinking Lemma,

$$B_{g(s)}(p_k, \frac{1}{2} r) \subset B_{g(0)}(p_k, \frac{1}{2} r + \beta\sqrt{c_0 T}) \subset B_{g(0)}(p_k, \frac{3}{4} r) \tag{4.16}$$

for each $k$ and $s \in [0, T]$. Thus for all $s \in [0, T]$,

$$B_{g(0)}(x_0, R - r) \cap \bigcup_{k=1}^{N} B_{g(s)}(p_k, \frac{1}{2} r) = \emptyset. \tag{4.17}$$

Then the second inclusion in (4.3) follows immediately from (4.17) and the definitions of $f$ and $\phi$. □

## 5. Heat kernel estimates for Ricci flow in expansion

### *5.1. An upper bound for the heat kernel of Ricci flow*

Let $(M, g(t))$, $t \in [0, T]$, be a complete Ricci flow. Hereafter we denote by $G(x, t; y, s)$, with $x, y \in M$, $0 \leq s < t \leq T$, the heat kernel corresponding to the backwards heat equation coupled with the Ricci flow. This means that for any fixed $(x, t) \in M \times [0, T]$ we have

$$(\frac{\partial}{\partial s} + \Delta_{y,s}) G(x, t; y, s) = 0 \text{ and } \lim_{s \nearrow t} G(x, t; y, s) = \delta_x(y) \tag{5.1}$$

Then for any fixed $(y,s) \in M \times [0,T]$ we can compute that $G(\cdot,\cdot;y,s)$ is the heat kernel associated to the conjugate equation

$$(\frac{\partial}{\partial t} - \Delta_{x,t} - \mathrm{scal}_{g(t)})G(x,t;y,s) = 0 \text{ and } \lim_{t \searrow s} G(x,t;y,s) = \delta_y(x). \tag{5.2}$$

Note that in literatures it is more common to consider the fundamental solution of the conjugate heat equation $\frac{\partial}{\partial t}u + \Delta_{x,t}u - \mathrm{scal}u = 0$. $G(x,t;y,s)$ has the following property

$$\int_M G(x,t;y,s)\, d_t x = 1 \text{ for all } 0 \le s < t \le T. \tag{5.3}$$

In the compact case, this follows from the following simple calculation:

$$\begin{aligned} &\frac{\partial}{\partial t} \int_M G(x,t;y,s)\, d_t x \\ &= \int_M ((\Delta_{x,t} + \mathrm{scal}_{g(t)})G(x,t;y,s) - G(x,t;y,s)\,\mathrm{scal}_{g(t)})\, d_t x = 0. \end{aligned} \tag{5.4}$$

The general case follows using an exhaustion and limiting argument.

The heat kernel $G$ has a Gaussian bound by the following proposition from [1].

**Proposition 5.1.** *Given $n \in \mathbb{N}$ and $A > 0$, there is a constant $C = C(n,A) < \infty$ such that the following holds: Let $(M^n, g(t))$, $t \in [0,T]$, be a complete Ricci flow satisfying*

$$|Rm|_{g(t)} \le \frac{A}{t} \quad \text{and} \quad Vol_{g(t)} B_{g(t)}(x,\sqrt{t}) \ge \frac{(\sqrt{t})^n}{A} \tag{5.5}$$

*for all $(x,t) \in M \times (0,T]$. Then*

$$G(x,t;y,s) \le \frac{C}{(t-s)^{\frac{n}{2}}} exp(-\frac{d_s^2(x,y)}{C(t-s)}) \text{ for all } 0 \le s < t \le T. \tag{5.6}$$

**Remark 5.2.** We note that (5.5) is invariant under rescaling and time shifting in the sense that for the Ricci flow $\hat{g}(\tau) = \frac{1}{t-s} g(\tau(t-s)+s), \tau \in [0,1]$, where $0 \le s < t \le T$, the condition (5.5) still holds true. The right-hand side of the second bound in (5.5) may change by a controlled factor due to a volume comparison argument.

### *5.2. Generalized heat kernel of Ricci flow in expansion and its upper bound*

**Definition 5.3.** (Ricci flow in expansion) We say $(\{M_j\}_{j=1}^m, \{g_j(t)\}_{j=1}^m, \nu)$ is a Ricci flow in expansion, if for each $j$, $(M_j, g_j(t))$ is a complete Ricci flow defined on $[t_j, t_{j+1}]$ with $t_1 > 0$, $t_{j+1} = \nu t_j$ for $j \ge 1$, and $M_0 \supset M_1 \supset M_2 \supset ... \supset M_m$. Moreover, at each $t_{j+1}$ we have $g_{j+1}(t_{j+1}) \ge g_j(t_{j+1})$ everywhere on $M_{j+1}$.

We call each $t_j$ a expanding time. In the following discussion we will often need to distinguish metrics $g_{j-1}(t_j)$ and $g_j(t_j)$. Without ambiguity, we use $t_j^+$ whenever referring to any geometric quantity with respect to $g_j(t_j)$, and $t_j^-$ for $g_{j-1}(t_j)$ respectively. For example, $B_{t_j^+}(x,r)$ denotes a $r$-ball centered at $x$ with respect to $g_j(t_j)$ and $M_{t_j^+}$ denotes $M_j$.

**Definition 5.4.** (Generalized heat kernel) Let $(\{M_j\}_{j=1}^m, \{g_j(t)\}_{j=1}^m, \nu)$ be a Ricci flow in expansion. For any $x \in M_i$ and $t \in (t_i, t_{i+1}]$, we define the generalized heat kernel $G(x,t;\cdot,\cdot)$ as follows: First, $G(x,t;y,s)$ is the standard heat kernel for all $y \in M_i$ and $s \in [t_i, t)$. Next, suppose $G(x,t;z,s')$ has been defined for all $z \in M_j$ and $s' \in [t_j, t_{j+1})$ for some $j \le i$. Then for $y \in M_{j-1}$ and $s \in [t_{j-1}, t_j)$, we set

$$G(x,t;y,s) = \int\limits_{M_{t_j^+}} G(x,t;z,t_j)G(z,t_j;y,s)d_{t_j^-}z. \tag{5.7}$$

Inductively, $G(x,t;\cdot,\cdot)$ is defined on $(\bigcup_{j=0}^{i-1} M_j \times [t_j, t_{j+1})) \cup M_i \times [t_i, t)$ (see Fig. 1). It's easy to see that $G(x,t;\cdot,\cdot)$ is continuous on all over its domain, and smooth on each $M_j \times (t_j, t_{j+1})$ for $j \le i-1$ and on $M_i \times (t_i, t)$.

The goal of this section is to derive a Gaussian bound for the generalized heat kernel. A crucial fact in the proof is that the $L^1$-norm of $G(\cdot, t_j; y, t_{j-1})$ is not bigger than 1 for all $t$, that is,

$$\int\limits_{M_{t_j^+}} G(x,t_j;y,t_{j-1})\, d_{t_j^-}x \le \int\limits_{M_{t_j^-}} G(x,t_j;y,t_{j-1})\, d_{t_j^-}x = 1 \tag{5.8}$$

for any $y \in M_{j-1}$.

**Proposition 5.5.** *Given $n \in \mathbb{N}$, $A > 0$, and $\nu > 1$, there is a constant $C = C(n, A, \nu) < \infty$ such that the following holds: Let $(\{M_j\}_{j=1}^m, \{g_j(t)\}_{j=1}^m, \nu)$ be a Ricci flow in expansion such that for each $j$ we have*

$$|Rm|_{g_j(t)} \le \frac{A}{t} \text{ and } Vol_{g_j(t)} B_{g_j(t)}(x, \sqrt{t}) \ge \frac{t^{\frac{n}{2}}}{A} \tag{5.9}$$

*for all $x \in M_j$ and $t \in [t_j, t_{j+1}]$. Then for any pairs $(x,t)$ and $(y,s)$ such that $G(x,t;y,s)$ is well defined as above, we have*

$$G(x,t;y,s) \le \frac{C}{(t-s)^{\frac{n}{2}}} exp\left(-\frac{d_{s^+}^2(x,y)}{C(t-s)}\right). \tag{5.10}$$

**Remark 5.6.** It may seem surprising that it is not necessary to assume the equality of metrics $g_j(t_{j+1})$ and $g_{j+1}(t_{j+1})$ on $M_{j+1}$. But as we will see in the proof below, the

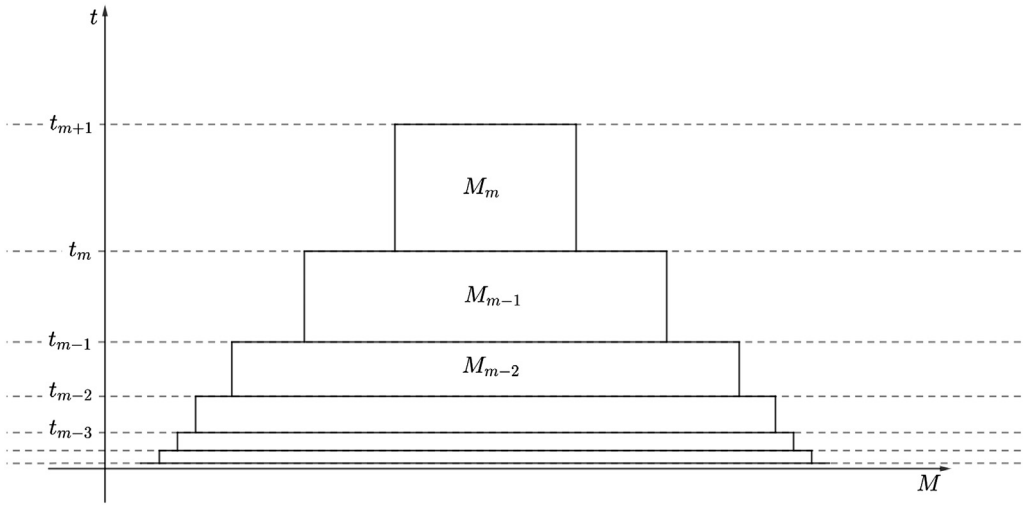


**Fig. 1.** Ricci flow in expansion.

expanding condition $g_j(t_{j+1}) \leq g_{j+1}(t_{j+1})$ is compatible with the application of the Shrinking Lemma and hence sufficient for us to get the conclusion. In later application to the proof of Theorem 1.1, the metric $g_{j+1}(t_{j+1})$ is the conformally changed metric of $g_j(t_{j+1})$, which is not less than $g_j(t_{j+1})$ everywhere on $M_{j+1}$, and agrees with it on a smaller region.

**Proof.** For notational convenience, the same letter $C$ will be used to denote constants depending on $n$, $A$ and $\nu$.

**Part 1** Let us first establish the estimate (5.10) for $t = t_{k+i}$ and $s = t_i$ for some $i \geq 1$ and $k \geq 1$. Rescaling the flow $g(t), t \in [t_i, t_{k+i}]$ to $\hat{g}(\tau) = \frac{1}{t_{k+i}-t_i} g(\tau(t_{k+i} - t_i) + t_i), \tau \in [0, 1]$, the "expanding time" sequence

$$t_{k+i} > t_{k+i-1} > \cdots > t_{k+i-j} > \cdots > t_{i+1} > t_i$$

becomes

$$1 = \tau_0 > \tau_1 > \cdots > \tau_j > \cdots > \tau_{k-1} > \tau_k = 0$$

where $\tau_j := \frac{t_{k+i-j}-t_i}{t_{k+i}-t_i} = \frac{\nu^{k-j}-1}{\nu^k-1}$, for $j = 0, 1, 2, ..., k$. Then for each $j = 0, 1, ..., k-1$, we have

$$\tau_j - \tau_{j+1} = \frac{\nu^{k-j} - \nu^{k-j-1}}{\nu^k - 1} \leq \nu^{-j}. \tag{5.11}$$

To show (5.10) for $t = t_{k+i}$ and $s = t_i$, it's equivalent to show the following inequality under the new flow:

$$G(x,1;y,0) \leq Cexp\left(-\frac{d_{0^+}^2(x,y)}{C}\right). \tag{5.12}$$

We note that by Remark 5.2, the new flow $\hat{g}(\tau)$ satisfies the curvature and volume conditions in (5.9).

Since $\tau_1 \leq \nu^{-1}$, applying the Gaussian bound (5.6) for standard heat kernel we find that

$$G(x,1;\cdot,\tau_1) \leq \frac{C}{(1-\tau_1)^{\frac{n}{2}}} \leq C_0 := \frac{C}{(1-\nu^{-1})^{\frac{n}{2}}}. \tag{5.13}$$

Let $C_0$ be fixed hereafter. Suppose by induction that $G(x,1;\cdot,\tau_j) \leq C_0$ for some $j \geq 1$. Then for any $z$ such that $G(x,1;z,\tau_{j+1})$ is well defined, we have

$$\begin{aligned} G(x,1;z,\tau_{j+1}) &= \int\limits_{M_{\tau_j^+}} G(x,1;w,\tau_j)G(w,\tau_j;z,\tau_{j+1})d_{\tau_j^-}w \\ &\leq C_0 \int\limits_{M_{\tau_j^+}} G(w,\tau_j;z,\tau_{j+1})d_{\tau_j^-}w \leq C_0 \end{aligned} \tag{5.14}$$

where we used (5.8) in the last inequality. So by induction we obtain

$$G(x,1;\cdot,\tau_j) \leq C_0, \tag{5.15}$$

for all $j = 1,2,...,k$. In particular, we have $G(x,1;\cdot,0) \leq C_0$. This implies (5.12) when $d_{0^+}(x,y)$ is controlled. So it remains to show $G(x,1;y,0) \leq \ exp\left(-\frac{d^2}{C}\right)$ whenever $d_{0^+}(x,y) \geq 4d(1-(\sqrt[4]{\nu})^{-1})$ for a large number $d$ (which we will specify in the course of proof). For each $j = 1,2,...,k$, let

$$r_j = 4d(1-(\sqrt[4]{\nu})^{-j}). \tag{5.16}$$

Then set $B_j = B_{\tau_j^+}(x,r_j)$, $C_j = M_{\tau_j^+} - B_j$ and

$$a_j := \sup_{C_j} G(x,1;\cdot,\tau_j). \tag{5.17}$$

Then it suffices to show the following Claim:

**Claim 5.7.** *$a_j \leq Cexp(-\frac{d^2}{C})$, for some constant $C$ independent of $d$, which is uniform for all $j = 1,2,...,k$.*

**Proof of Claim 5.7.** For each $j$, the expanding condition $g_{j-1}(t_j) \leq g_j(t_j)$ implies $B_j = B_{\tau_j^+}(x,r_j) \subset B_{\tau_j^-}(x,r_j)$. Applying the Shrinking Lemma on $[\tau_{j+1},\tau_j]$, we find

that $B_{\tau_j^-}(x, r_j) \subset B_{\tau_{j+1}^+}(x, r_j + \beta\sqrt{A}\sqrt{\tau_j - \tau_{j+1}})$. Thus for any $z \in C_{j+1}$ and $w \in B_j$, the triangle inequality implies

$$d_{\tau_{j+1}^+}(z, w) \geq r_{j+1} - r_j - \beta\sqrt{A}\sqrt{\tau_j - \tau_{j+1}}. \tag{5.18}$$

By (5.11), $\sqrt{\tau_j - \tau_{j+1}} \leq (\sqrt{\nu})^{-j} \leq (\sqrt[4]{\nu})^{-j}$, we choose

$$d \geq \frac{\beta\sqrt{A}}{2(1 - (\sqrt[4]{\nu})^{-1})},$$

then (5.18) gives

$$d_{\tau_{j+1}^+}(z, w) \geq \delta r_j := \frac{2d(1 - (\sqrt[4]{\nu})^{-1})}{(\sqrt[4]{\nu})^j}. \tag{5.19}$$

To conclude, we have

$$B_j \subset M_{\tau_{j+1}^+} - B_{\tau_{j+1}^+}(z, \delta r_j). \tag{5.20}$$

By Definition 5.4, we have

$$G(x, 1; z, \tau_{j+1}) = \int\limits_{M_{\tau_j^+}} G(x, 1; w, \tau_j) G(w, \tau_j; z, \tau_{j+1})\, d_{\tau_j^-} w, \tag{5.21}$$

for any $z \in C_{j+1}$ fixed. We split the following integral $\mathcal{I}[M_{\tau_j^+}] := G(x, 1; z, \tau_{j+1})$ into the integrals over $C_j$ and $B_j$. We obtain from the definition of $a_j$ and (5.8) that

$$\begin{aligned} \mathcal{I}[C_j] &= \int\limits_{C_j} G(x, 1; w, \tau_j) G(w, \tau_j; z, \tau_{j+1}) d_{\tau_j^-} w \\ &\leq a_j \int\limits_{M_{\tau_j^+}} G(w, \tau_j; z, \tau_{j+1}) d_{\tau_j^-} w \leq a_j. \end{aligned} \tag{5.22}$$

To estimate $\mathcal{I}[B_j]$, we notice that by (5.20), (5.15) and (5.9) we have

$$\begin{aligned} \mathcal{I}[B_j] &\leq C_0 \int\limits_{B_j} G(w, \tau_j; z, \tau_{j+1})\ d_{\tau_j^-} w, \\ &\leq C_0 \int\limits_{M_{\tau_{j+1}^+} - B_{\tau_{j+1}^+}(z, \delta r_j)} G(w, \tau_j; z, \tau_{j+1})\ d_{\tau_j^-} w \\ &\leq C \int\limits_{M_{\tau_{j+1}^+} - B_{\tau_{j+1}^+}(z, \delta r_j)} G(w, \tau_j; z, \tau_{j+1})\ d_{\tau_{j+1}^+} w. \end{aligned}$$

Then applying the Gaussian bound (5.6) to $G(w, \tau_j; z, \tau_{j+1})$ and calculating as in Lemma 2.8 we have

$$\mathcal{I}[B_j] \leq Cexp\left(-\frac{(\delta r_j)^2}{C(\tau_j - \tau_{j+1})}\right). \tag{5.23}$$

Plugging (5.11) and (5.19) into (5.23) we have

$$\mathcal{I}[B_j] \leq Cexp\left(-\frac{(\sqrt{\nu})^j d^2}{C}\right). \tag{5.24}$$

Combining (5.22) and (5.23), we see that $G(x, 1; z, \tau_{j+1}) \leq a_j + Cexp(-\frac{d^2(\sqrt{\nu}^j)}{C})$ for arbitrary $z$ in $C_{j+1}$. Hence by the definition of $a_{j+1}$, there holds

$$\begin{aligned} a_{j+1} &\leq a_j + Cexp\left(-\frac{d^2(\sqrt{\nu})^j}{C}\right) \leq a_1 + C\sum_{l=1}^{j} exp\left(-\frac{d^2(\sqrt{\nu})^l}{C}\right) \\ &\leq a_1 + Cexp\left(-\frac{d^2}{C}\right)\sum_{l=1}^{j} exp\left(-\frac{d^2((\sqrt{\nu})^l - 1)}{C}\right) \\ &\leq a_1 + C\,exp\left(-\frac{d^2}{C}\right). \end{aligned}$$

Note $a_1 = \sup_{C_1} G(x, 1; \cdot, \tau_1)$. For any $z \in C_1 = M_{\tau_1^+} - B_{\tau_1^+}(x, r_1)$, we have $d_{\tau_1^+}(x, z) \geq r_1 = 4d(1 - (\sqrt[4]{\nu})^{-1})$. Substituting this into the ordinary Gaussian bound, we get $G(x, 1; z, \tau_1) \leq \frac{C}{(1-\tau_1)^{\frac{n}{2}}} exp(-\frac{d^2}{C(1-\tau_1)})$. This gives $a_1 \leq Cexp(-\frac{d^2}{C})$. Hence $a_{j+1} \leq Cexp(-\frac{d^2}{C})$. This finishes the proof of the claim. □

To summarize, we showed for $t = t_{k+i}, s = t_i$, $i \geq 1, k \geq 1$ and $x, y$ such that $G(x, t_{k+i}; y, t_i)$ is defined, we have the Gaussian bound.

$$G(x, t_{k+i}; y, t_i) \leq \frac{C}{(t_{k+i} - t_i)^{\frac{n}{2}}} exp\left(-\frac{d^2_{t_i^+}(x, y)}{C(t_{k+i} - t_i)}\right). \tag{5.25}$$

We will use this to derive the Gaussian bound (5.10) for arbitrary $t$ and $s$.

**Part 2** To show (5.10) for arbitrary $t$ and $s$, there are two cases left. The first is that neither $t$ nor $s$ is an expanding time, and the second is that one of them is an expanding time. Since the second case follows a same but easier route than the first one, we prove the first case below.

Since $t$ and $s$ are not expanding times, we may assume $t \in (t_{k+i}, t_{k+i+1})$ and $s \in (t_i, t_{i+1})$ for some $k$ and $i$. Rescaling the flow on $[s, t]$ to a new flow on $[0, 1]$, for the same reason as in Part 1, it suffices to show for any very large $d$ (which we specify below) and $x, y$ such that $d_0(x, y) \geq 5d$, we have

$$G(x,1;y,0) \leq C exp\left(-\frac{d^2}{C}\right). \tag{5.26}$$

Under rescaling, $t_{k+i}$ and $t_{i+1}$ become $\tau_2 := \frac{t_{k+i}-s}{t-s}$ and $\tau_1 := \frac{t_{i+1}-s}{t-s}$, respectively. By Definition 5.4 of the generalized heat kernel, we have

$$G(x,1;y,0) = \int\limits_{M_{\tau_2^+}} \int\limits_{M_{\tau_1^+}} G(x,1;z,\tau_2)G(z,\tau_2;w,\tau_1)G(w,\tau_1;y,0)d_{\tau_1^-}w\, d_{\tau_2^-}z. \tag{5.27}$$

We split the integral $\mathcal{I}[M_{\tau_2^+} \times M_{\tau_1^+}] := G(x,1;y,0)$ over three regions

$$\begin{aligned} U &= \{(z,w) \,|\, z \in B_{\tau_2^-}(x,d) \text{ and } w \in B_{\tau_1^-}(y,d)\}, \\ V &= \{(z,w) \,|\, z \notin B_{\tau_2^-}(x,d)\}, \\ W &= \{(z,w) \,|\, w \notin B_{\tau_1^-}(y,d)\}. \end{aligned} \tag{5.28}$$

Then $G(x,1;y,0) \leq \mathcal{I}[U] + \mathcal{I}[V] + \mathcal{I}[W]$. Since $\tau_1$ and $\tau_2$ are both expanding times and $\tau_2 - \tau_1$ is bounded below by a positive number depending only on $\nu$, the result from Part 1 implies

$$G(z,\tau_2;w,\tau_1) \leq C\, exp\left(-\frac{d^2_{\tau_1^+}(z,w)}{C}\right). \tag{5.29}$$

If we choose $d \geq \beta\sqrt{A}$, then for any $z \in B_{\tau_2^-}(x,d)$ and $w \in B_{\tau_1^-}(y,d)$, the Shrinking Lemma together with the expanding conditions imply $d_{\tau_1^-}(x,z) \leq d_{\tau_1^+}(x,z) \leq d_{\tau_2^-}(x,z) + \beta\sqrt{A} \leq d_{\tau_2^-}(x,z) + d \leq 2d$, and $d_{\tau_1^-}(x,y) \geq d_0(x,y) - \beta\sqrt{A} \geq 4d$. Then by triangle inequality we have

$$d_{\tau_1^+}(z,w) \geq d_{\tau_1^-}(z,w) \geq d_{\tau_1^-}(x,y) - d_{\tau_1^-}(x,z) - d_{\tau_1^-}(y,w) \geq d. \tag{5.30}$$

Hence by (5.29), (5.30), (5.6) and (5.8) we have

$$\begin{aligned} \mathcal{I}[U] &\leq C\, exp\left(-\frac{d^2}{C}\right) \cdot \left(\int\limits_{M_{\tau_2^+}} G(x,1;z,\tau_2)\, d_{\tau_2^-}z\right) \cdot \left(\int\limits_{M_{\tau_1^+}} G(w,\tau_1;y,0)\, d_{\tau_1^-}w\right) \\ &\leq C\, exp\left(-\frac{d^2}{C}\right) \cdot C \cdot 1 = C\, exp\left(-\frac{d^2}{C}\right). \end{aligned} \tag{5.31}$$

And (5.29), (5.8), (5.6) together with Lemma 2.8 imply

$$\mathcal{I}[V] \leq C \left( \int\limits_{z \notin B_{\tau_2^-}(x,d)} G(x,1;z,\tau_2)\ d_{\tau_2^-} z \right) \leq C exp\left(-\frac{d^2}{C}\right). \tag{5.32}$$

Similarly we have

$$\mathcal{I}[W] \leq C exp\left(-\frac{d^2}{C}\right). \tag{5.33}$$

So (5.26) follows from (5.31), (5.32), (5.33) immediately. □

### 5.3. Gradient of heat kernel

In this subsection, we consider Ricci flow in expansion $(\{M_j\}_{j=1}^m, \{g_j(t)\}_{j=1}^m, \nu)$ and use Proposition 5.5 to derive an upper bound for the gradient of the generalized heat kernel. Assume all the conditions are the same as in Proposition 5.5. We choose and fix some $x \in M_i$, $t \in (t_i, t_{i+1}]$ for some $i$. Then $G(x,t;\cdot,\cdot)$ is a solution to the heat equation $\frac{\partial}{\partial s'} G(x,t;z,s') + \Delta_{z,s'} G(x,t;z,s') = 0$ on $M_j \times (t_j, \min(t_{j+1}, t)]$, $j = 1, ..., i$. For an arbitrary $(y,s) \in M_j \times (t_j, \min(t_{j+1}, t)]$, $j = 1, ..., i$, applying the standard result of Schauder estimate (see [8] for example), we see that there is a constant $C$ depending on $A$ and $n$ such that

$$|\nabla G|(x,t;y,s) \leq \frac{C}{\sqrt{s-t_j}} \sup G(x,t;\cdot,\cdot), \tag{5.34}$$

where the supremum is taken over $B_{g(s)}(y, \sqrt{s-t_j}) \times [t_j, s]$.

Since $|\mathrm{Rm}| \leq \frac{A}{t}$ on $M_j \times [t_j, t_{j+1}]$, we have a constant $C_1 = C_1(n, A, \nu) > 0$ such that for any $s, s' \in [t_j, t_{j+1}]$, $C_1^{-1} d_{s'} \leq d_s \leq C_1\, d_{s'}$. Suppose $d_s(x,y) \geq d$ for a large number $d$ satisfying

$$d \geq 2C_1(\sqrt{t_{i+1}-t_i} + \beta\sqrt{A}\sqrt{t_{i+1}-t_i}). \tag{5.35}$$

We claim the following Gaussian bound of $|\nabla G|(x,t;y,s)$:

**Claim 5.8.** *For each $j = 1, ..., i$, we have the following estimate:*

$$|\nabla G|(x,t;y,s) \leq \frac{1}{\sqrt{s-t_j}} \frac{C}{t_{i+1}^{\frac{n}{2}}} exp\left(-\frac{d_s^2(x,y)}{Ct_{i+1}}\right) \tag{5.36}$$

*for some constant $C$ that only depends on $A$, $\nu$ and $n$.*

**Proof of Claim 5.8.** For any $(z, s') \in B_{g(s)}(y, \sqrt{s-t_j}) \times [t_j, s]$, first we have by the Shrinking Lemma that $d_{s'}(y,z) \le d_s(y,z) + \beta\sqrt{A}(\sqrt{s-s'})$. Then the triangle inequality and (5.35) we get

$$\begin{aligned} d_{s'}(x,z) &\ge d_{s'}(x,y) - d_{s'}(y,z) \\ &\ge d_{s'}(x,y) - d_s(y,z) - \beta\sqrt{A}(\sqrt{s-s'}) \\ &\ge d_{s'}(x,y) - \sqrt{t_{j+1}-t_j} - \beta\sqrt{A}\sqrt{t_{j+1}-t_j} \\ &\ge C_1^{-1} d_s(x,y) - \sqrt{t_{j+1}-t_j} - \beta\sqrt{A}\sqrt{t_{j+1}-t_j} \\ &\ge \tfrac{1}{2} C_1^{-1} d_s(x,y). \end{aligned} \tag{5.37}$$

So by Proposition 5.5 we have

$$G(x,t;z,s') \le \frac{C}{(t-s')^{\frac{n}{2}}} exp\left(-\frac{d_{s'}^2(x,z)}{C(t-s')}\right) \le \frac{C}{(t-s')^{\frac{n}{2}}} exp\left(-\frac{d_s^2(x,y)}{C(t-s')}\right). \tag{5.38}$$

Since $d_s(x,y) \ge d$ and $t - s' \le t_{i+1}$, Lemma 2.9 implies

$$G(x,t;z,s') \le \frac{C}{(t_{i+1})^{\frac{n}{2}}} exp\left(-\frac{d_s^2(x,y)}{Ct_{i+1}}\right). \tag{5.39}$$

The claim thus follows by letting $(z, s')$ run over $B_{g(s)}(y, \sqrt{s-t_j}) \times [t_j, s]$. □

## 6. Proof of Theorem 1.1

First, we consider the conditions given in Theorem 1.1. The upper bound on $\ell(x,0)$ implies a lower bound on Ricci curvature, that is, $\ell(x,0) \le \alpha_0 \le 1$ implies $\mathrm{Ric} \ge -K(n)$. So by Bishop–Gromov comparison, reducing $v_0$ to a smaller positive number depending only on the original $v_0$ and $n$, we may assume without loss of generality that

$$Vol_{g(0)} B_{g(0)}(x,r) \ge v_0 r^n \tag{6.1}$$

for all $x \in B_{g(0)}(x_0, s_0 - 1)$ and $r \in (0,1]$. We can also assume $\alpha_0$ without loss of generality that

$$\alpha_0 \le \frac{1}{2C_4} < 1 \tag{6.2}$$

where $C_4 = C_4(v_0, n) > 2$ is to be determined later. Otherwise, we get the result by applying the above result to a rescaled metric and then scale it back.

By the relative compactness of $B_{g(0)}(x_0, s_0)$, there exists some $\rho \in (0, \frac{1}{2}]$ such that $|\mathrm{Rm}| \le \frac{1}{\rho^2}$, $B_{g(0)}(x,\rho) \subset\subset M$ and $\mathrm{inj}_{g(0)}(x) \ge \rho$ for all $x \in B_{g(0)}(x_0, s_0)$. The constant $\rho$ may depend on $(M, g(0))$, $x_0$ and $s_0$. By applying Lemma 2.6, with $U := B_{g(0)}(x_0, s_0)$, we can find a connected subset $\tilde{M} \subset U \subset M$ containing $B_{g(0)}(x_0, s_0 - \frac{1}{2})$, and a smooth,

complete metric $\tilde{g}(0)$ on $\tilde{M}$ with $\sup_{\tilde{M}} |\mathrm{Rm}|_{\tilde{g}(0)} < \infty$ such that on $B_{g(0)}(x_0, r_0)$, where $r_0 := s_0 - 1 > 3$, the metric remains unchanged. Taking Shi's Ricci flow we get a smooth, complete, bounded-curvature Ricci flow $g_0(t)$ on $M_0 := \tilde{M}$, existing for some nontrivial time interval $[0, t_1]$. In view of the boundedness of the curvature, after possibly reducing $t_1$ to a smaller positive value, we may trivially assume that $|\mathrm{Rm}|_{g(t)} \leq \frac{C_3}{t}$ for all $t \in (0, t_1]$ and $\ell(x,t) \leq 2\alpha_0 < 1$ for all $x \in B_{g(0)}(x_0, r_0)$ and $t \in [0, t_1]$. The constant $C_3 = C_3(v_0, n)$ will be given below.

Of course, our flow still lacks a uniform control on its existence time. Below we will carry out an inductive argument to show that $t_1$ could be extended up to a uniform time $t_k$, while the repeating time $k$ may be allowed to depend on $(M, g)$.

Now we begin the proof of Theorem 1.1. First, suppose we have constructed a Ricci flow in expansion $(\{M_j\}_{j=1}^i, \{g_j(t)\}_{j=1}^i, \nu)$ with $(M_0, g_0(t))_{t\in[0,t_1]}$ as above. Suppose further the Ricci flow in expansion satisfies the following a priori assumptions:

**(APA 1)** Restricting it on $B_{g(0)}(x_0, r_i)$, we get a smooth Ricci flow $g(t)$ up to $t_{i+1}$;
**(APA 2)** For each complete Ricci flow $(M_j, g_j(t))$, we have $|\mathrm{Rm}|_{g_j(t)} \leq \frac{C_3}{t}$;
**(APA 3)** $\ell(x,t) \leq C_4\alpha_0 < 1$ for all $t \in [0, t_{i+1}]$ and $x \in B_{g(0)}(x_0, r_i)$,

where the constants $C_3, C_4, \nu$ depending on $v_0, n$ will be specified in the course of the proof.

Our goal is to extend it to a new Ricci flow in expansion $(\{M_j\}_{j=1}^{i+1}, \{g_j(t)\}_{j=1}^{i+1}, \nu)$ by adding a complete Ricci flow $(M_{i+1}, g_{i+1}(t))$ piece existing for $[t_{i+1}, t_{i+2}]$, and show that it still satisfies (APA 1)–(APA 3). In the current section, we construct $(M_{i+1}, g_{i+1}(t))$, and then verify (APA 1) and (APA 2), and we leave the verification of (APA 3) to the next section.

Let $C_1 \geq 1$ and $\tilde{T} > 0$ be the constants from the Curvature Decay Lemma (Lemma 3.4) when $K = 1$ and $v_0 = v_0$. With this choice of $C_1$, we set $C_2 = \gamma C_1$ and $C_3 = 4C_2 = 4\gamma C_1 > 1$, where $\gamma = \gamma(n) \geq 1$ is the constant from the Conformal Change Lemma (Lemma 2.6), and set $\nu = 1 + \frac{1}{4C_3}$. Choose $\tau$ such that

$$\tau \leq \hat{T}, \quad \beta^2 C_3 \tau \leq \frac{1}{16}\sqrt{\tau} \leq 1, \quad \tau \leq \frac{1}{16}, \quad \tau \leq \frac{C_1}{4}, \tag{6.3}$$

where $\beta \geq 1$ is the constant from the Shrinking Lemma. We can also assume that $2t_{i+1} \leq \tau$, because otherwise we get the desired uniform existence time $\frac{\tau}{2}$.

In the Claim below, we show that in fact we have a stronger curvature decay bound $|\mathrm{Rm}|_{g(t)} \leq \frac{C_1}{t}$. However, the original curvature decay will nevertheless be used to control the distance distortion.

**Claim 6.1.** *For all $x \in U := B_{g(0)}(x_0, r_i - 2\sqrt{\frac{t_{i+1}}{\tau}})$, we have $B_{g(t)}(x, \sqrt{\frac{t}{\tau}}) \subset\subset B_{g(0)}(x_0, r_i)$, $inj_{g(t)}(x) \geq \sqrt{\frac{t}{C_1}}$ and $|Rm|_{g(t)}(x) \leq \frac{C_1}{t}$, for all $t \in (0, t_{i+1}]$.*

**Proof of Claim 6.1.** For any $x \in B_{g(0)}(x_0, r_i - 2\sqrt{\frac{t_{i+1}}{\tau}})$, the triangle inequality implies that $B_{g(0)}(x, 2\sqrt{\frac{t_{i+1}}{\tau}}) \subset\subset B_{g(0)}(x_0, r_i)$ and hence by assumption (APA 3), $\ell(y,t) \leq 1$ on $B_{g(0)}(x, 2\sqrt{\frac{t_{i+1}}{\tau}})$ for all $t \in [0, t_{i+1}]$. Scaling the solution to $\hat{g}(t) := \frac{\tau}{t_{i+1}} g(t\frac{t_{i+1}}{\tau})$ we see that we have a solution $\hat{g}(t)$ on $B_{g(0)}(x_0, r_i) \supset\supset B_{\hat{g}(0)}(x, 2)$, $t \in [0, \tau]$ with $|\mathrm{Rm}|_{\hat{g}(t)} \leq \frac{C_3}{t}$ and $\ell(\cdot,\cdot) \leq 1$ on $B_{\hat{g}(0)}(x, 2) \times (0, \tau]$.

On the one hand, applying the Shrinking Lemma to $\hat{g}(t)$, we find that $B_{\hat{g}(t)}(x, 2 - \beta\sqrt{C_3 t}) \subset B_{\hat{g}(0)}(x, 2)$ for all $t \in [0, \tau]$, and in particular $B_{\hat{g}(t)}(x, 1) \subset B_{\hat{g}(0)}(x, 2)$ because $\tau \leq \frac{1}{\beta^2 C_3}$. Thus we have $\ell(\cdot,\cdot) \leq 1$ on $\bigcup_{s\in[0,\tau]} B_{\hat{g}(s)}(x, 1) \times [0, \tau]$. On the other hand, the volume inequality (6.1) transforms to $Vol_{\hat{g}(0)} B_{\hat{g}(0)}(x, 1) \geq v_0$.

Applying the Curvature Decay Lemma (Lemma 3.4) to $\hat{g}(t)$, we have $\mathrm{inj}_{\hat{g}(t)}(x) \geq \sqrt{\frac{t}{C_1}}$ and $|\mathrm{Rm}|_{\hat{g}(t)}(x) \leq \frac{C_1}{t}$ for all $0 < t \leq \tau$. Scaling back, we see that $B_{g(t)}(x, \sqrt{\frac{t_{i+1}}{\tau}}) \subset\subset B_{g(0)}(x_0, r_i)$, $\mathrm{inj}_{g(t)}(x) \geq \sqrt{\frac{t}{C_1}}$ and $|\mathrm{Rm}|_{g(t)}(x) \leq \frac{C_1}{t}$ for $t \in (0, t_{i+1}]$. □

Specializing the Claim 6.1 to $t = t_{i+1}$, we have $|\mathrm{Rm}|_{g(t_{i+1})}(x) \leq \frac{C_1}{t_{i+1}}$ and $\mathrm{inj}_{g(t_{i+1})}(x) \geq \sqrt{\frac{t_{i+1}}{C_1}}$ for any $x \in U := B_{g(0)}(x_0, r_i - 2\sqrt{\frac{t_{i+1}}{\tau}})$. Now we apply the Conformal Change Lemma 2.6 with $U = B_{g(0)}(x_0, r_i - 2\sqrt{\frac{t_{i+1}}{\tau}})$, $N = B_{g(0)}(x_0, r_i)$, $g(t_{i+1})$ and $\rho^2 := \frac{t_{i+1}}{C_1} \leq 1$, and obtain a new, possibly disconnected, smooth manifold $(\tilde{U}, h)$, each component of which is complete, such that

1. $|\mathrm{Rm}|_h \leq \gamma \frac{C_1}{t_{i+1}} = \frac{C_2}{t_{i+1}}$ and $\mathrm{inj}_h \geq \sqrt{\frac{t_{i+1}}{\gamma C_1}} = \sqrt{\frac{t_{i+1}}{C_2}}$ for all $x \in \tilde{U}$,
2. $U_\rho \subset \tilde{U} \subset U$,
3. $h = g(t_{i+1})$ on $\tilde{U}_\rho \supset U_{2\rho}$

where $U_r = \{x \in U | B_g(x, r) \subset\subset U\}$.

**Claim 6.2.** *We have $B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}}) \subset U_{2\rho}$ where the metric $g(t_{i+1})$ and $h$ agree.*

**Proof of Claim 6.2.** By definition of $U$, for every $x \in B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}})$, the triangle inequality implies $B_{g(0)}(x, 2\sqrt{\frac{t_{i+1}}{\tau}}) \subset\subset U$. By (APA 2), we have $|\mathrm{Rm}|_{g(t)} \leq \frac{C_3}{t}$ on $B_{g(0)}(x_0, r_i)$, and hence on $B_{g(0)}(x, 2\sqrt{\frac{t_{i+1}}{\tau}})$ for all $t \in (0, t_{i+1}]$. Applying the Shrinking Lemma we have $B_{g(0)}(x, 2\sqrt{\frac{t_{i+1}}{\tau}}) \supset B_{g(t)}(x, 2\sqrt{\frac{t_{i+1}}{\tau}} - \beta\sqrt{C_3 t})$ for all $t \in [0, t_{i+1}]$. Specializing to $t = t_{i+1}$ and use $\beta\sqrt{C_3 t_{i+1}} \leq \sqrt{\frac{t_{i+1}}{\tau}}$ we see that $B_{g(t_{i+1})}(x, \sqrt{\frac{t_{i+1}}{\tau}}) \subset\subset U$. By (6.3) this gives $B_{g(t_{i+1})}(x, 2\sqrt{\frac{t_{i+1}}{C_1}}) \subset\subset U$ which means $x \in U_{2\rho}$ by definition of $\rho$. □

In view of Claim 6.2 we define the connected component of $(\tilde{U}, h)$ that contains $B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}})$ as $M_{i+1}$. Then we restart the flow from $(M_{i+1}, h)$ using Shi's complete bounded curvature Ricci flow. By the doubling time estimate (Lemma 1.4), we have a complete Ricci flow $(M_{i+1}, h(t))$ with $h(0) = h$ existing for $t \in [0, (\nu - 1)t_{i+1}]$ and satisfying

$$|\mathrm{Rm}|_{h(t)}(y) \leq 2\frac{C_2}{t_{i+1}} \quad \text{and} \quad Vol_{h(t)}B_{h(t)}(y, \sqrt{t_{i+1}}) \geq \frac{t_{i+1}^{\frac{n}{2}}}{A_0} \tag{6.4}$$

for all $y \in M_{i+1}$, where $A_0$ is a constant depending on $C_2$ and thus on $v_0$ and $n$. Setting $g_{i+1}(t) = h(t - t_{i+1})$ for $t \in [t_{i+1}, t_{i+2}] = [t_{i+1}, \nu t_{i+1}]$, we obtain a new Ricci flow in expansion $(\{M_j\}_{j=1}^{i+1}, \{g_j(t)\}_{j=1}^{i+1}, \nu)$, which clearly satisfies (APA 1). By (6.4) and $t_{i+2} = \nu t_{i+1}$ we have

$$|\mathrm{Rm}|_{g(t)}(y) \leq 2\frac{C_2}{t_{i+1}} \leq \frac{C_3}{t} \tag{6.5}$$

for all $t \in [t_{i+1}, t_{i+2}]$. Hence we verified (APA 2). For the same reason, we have

$$Vol_{g(t)}B_{g(t)}(y, \sqrt{t}) \geq \frac{t_{i+1}^{\frac{n}{2}}}{A_0} \geq \frac{t^{\frac{n}{2}}}{A} \tag{6.6}$$

for all $t \in [t_{i+1}, t_{i+2}]$, where $A = A_0\nu^{\frac{n}{2}}$ also depends on $v_0$ and $n$. The volume estimate is needed to apply Proposition 5.5 in next section.

## 7. Induction step: verification of (APA 3)

In this section we finish the proof of Theorem 1.1 by verifying (APA 3) for $(\{M_j\}_{j=1}^{i+1}, \{g_j(t)\}_{j=1}^{i+1}, \nu)$. More specifically, we determine $r_{i+1}$ such that when restricted on $B_{g(0)}(x_0, r_{i+1})$, the smooth Ricci flow $g(t)$ satisfies $\ell(x,t) \leq C_4\alpha_0 < 1$ for all $t \in [0, t_{i+2}]$. The estimates (6.5) and (6.6) allow us to apply Proposition 5.5 to $(\{M_j\}_{j=1}^{i+1}, \{g_j(t)\}_{j=1}^{i+1}, \nu)$, and get the Gaussian bound for the generalized heat kernel $G(x, t; y, s)$:

$$G(x, t; y, s) \leq \frac{C}{(t-s)^{\frac{n}{2}}} exp(-\frac{d_{s^+}^2(x,y)}{C(t-s)}), \tag{7.1}$$

where $C$ depends on $v_0$ and $n$. We will frequently use this inequality implicitly in this section. Also for notational convenience, the same letter $C$ will be used to denote positive constants depending on $n$ and $v_0$. We divide the integration estimates of $\ell$ into two steps.

**Step 1** We derive a rough bound for $\ell$. Specifically, we show that $\ell$ is bounded above by a constant depending only on $v_0$ and $n$. This bound gives a lower bound for Ricci curvature with the same dependence, which will be used in the second step.

**Claim 7.1.** *For any* $(x,t) \in B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}} - \sqrt[4]{t_{i+2}}) \times [0, t_{i+2}]$*, we have* $\ell(x,t) \le C$ *and correspondingly* $Ric \ge -K$*, where both* $C$ *and* $K$ *are positive constants depending only on* $v_0$ *and* $n$*.*

**Proof.** Since $(\{M_j\}_{j=1}^i, \{g_j(t)\}_{j=1}^i, \nu)$ satisfies (APA 3), we have $\ell(\cdot,\cdot) \le 1$ on $B_{g(0)}(x_0, r_i) \times [0, t_{i+1}]$. Thus it only remains to show $\ell(x,t) \le C$ for $t \in [t_{i+1}, t_{i+2}]$. Recall the evolution inequality of $\ell$.

$$\frac{\partial}{\partial t}\ell(x,t) \le \Delta\ell(x,t) + \mathrm{scal}(x,t)\ell(x,t) + C(n)\ell^2(x,t). \tag{7.2}$$

Using the curvature decay $|\mathrm{Rm}|_{g(t)} \le \frac{C_3}{t}$ we verified in Section 6, we have $\ell(x,t) \le \frac{C}{t_{i+1}}$ for all $(x,t) \in M_{i+1} \times [t_{i+1}, t_{i+2}]$. Substituting this into (7.2), we get

$$\frac{\partial}{\partial t}\ell \le \Delta\ell + \mathrm{scal}\,\ell + C\ell^2 \le \Delta\ell + \mathrm{scal}\ell + \frac{C}{t_{i+1}}\ell \tag{7.3}$$

in the barrier sense. For any $t \in [t_{i+1}, t_{i+2}]$, set $\mathcal{L}(x,t) = \ell(x,t)e^{-\frac{C}{t_{i+1}}t}$. Then $\mathcal{L}(x, t_{i+1}) = \ell(x, t_{i+1})e^{-C} \le e^{-C}$ and

$$\frac{\partial}{\partial t}\mathcal{L} \le \Delta\mathcal{L} + \mathrm{scal}\,\mathcal{L} \tag{7.4}$$

in the barrier sense. Let $h(x,t) = \int_{M_{i+1}} G(x,t;z,t_{i+1})\mathcal{L}(z,t_{i+1})d_{t_{i+1}}z$, then $h$ solves the following initial value problem:

$$\frac{\partial}{\partial s}h = \Delta h + \mathrm{scal}\,h \quad \text{and} \quad h(\cdot, t_{i+1}) = \mathcal{L}(\cdot, t_{i+1}). \tag{7.5}$$

By the maximum principle, we have

$$\mathcal{L}(x,t) \le h(x,t) = \mathcal{I}[M_{i+1}] := \int\limits_{M_{i+1}} G(x,t;y,t_{i+1})\mathcal{L}(y,t_{i+1})\, d_{t_{i+1}}y \tag{7.6}$$

for all $x \in M_{i+1}$ and $t \in [t_{i+1}, t_{i+2}]$.

Seeing that $B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}}) \subset M_{i+1}$ is where the smooth local flow exists up to $t_{i+2}$, we split the integral $\mathcal{I}[M_{i+1}]$ into two integrals over $\mathcal{B}_{i+1} := B_{g(0)}(x, \sqrt[4]{t_{i+2}})$ and $\mathcal{C}_{i+1} := M_{i+1} - B_{g(0)}(x, \sqrt[4]{t_{i+2}})$. Since $\mathcal{B}_{i+1} \subset B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}})$ where $\mathcal{L}(\cdot, t_{i+1}) \le \ell(\cdot, t_{i+1}) \le 1$ by (APA 3), we can estimate

$$\mathcal{I}[\mathcal{B}_{i+1}] \le \int\limits_{M_{i+1}} G(x,t;y,s)\, d_s y \le C. \tag{7.7}$$

To estimate $\mathcal{I}[\mathcal{C}_{i+1}]$, we first estimate $d_{t_{i+1}}(x,y)$ for any $y \in \mathcal{C}_{i+1}$ by the Shrinking Lemma and (6.3):

$$d_{t_{i+1}}(x,y) \geq \sqrt[4]{t_{i+2}} - \beta\sqrt{C_3}\sqrt{t_{i+1}} \geq \frac{1}{2}\sqrt[4]{t_{i+2}} \geq \sqrt{t_{i+2}} \geq \sqrt{t-t_{i+1}}. \tag{7.8}$$

Then by Lemma 2.9 we have

$$G(x,t;y,t_{i+1}) \leq \frac{C}{(t_{i+2})^{\frac{n}{2}}} exp(-\frac{d^2_{t_{i+1}}(x,y)}{C\,t_{i+2}}) \leq \frac{C}{(t_{i+2})^{\frac{n}{2}}} exp(-\frac{1}{C\sqrt{t_{i+2}}}). \tag{7.9}$$

Now we apply Lemma 2.8 at $t_{i+1}$, and combining with $\mathcal{L}(y,t_{i+1}) \leq \frac{C}{t_{i+1}}$ to obtain:

$$\mathcal{I}[\mathcal{C}_{i+1}] \leq C\,exp(-\frac{1}{C\sqrt{t_{i+2}}}) \leq C. \tag{7.10}$$

Hence Claim 7.1 follows by (7.7) and (7.10). □

**Step 2** It remains to convert this upper bound in Lemma 7.1 to the stronger upper bound as claimed in (APA 3). Using the bound for $\ell$ from Claim 7.1, we get the following linearization of the evolution equation for $\ell$ on $B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}} - \sqrt[4]{t_{i+2}}) \times [0, t_{i+2}]$:

$$\frac{\partial \ell}{\partial t} \leq \Delta\ell + \mathrm{scal}\ell + C(n)\ell^2 \leq \Delta\ell + \mathrm{scal}\ell + C\ell \tag{7.11}$$

in the barrier sense. Setting $\mathcal{L}(\cdot,t) = e^{-Ct}\ell(\cdot,t)$, we get $\frac{\partial}{\partial t}\mathcal{L} \leq \Delta\mathcal{L} + \mathrm{scal}\mathcal{L}$ on the same region as above, in the barrier sense.

Hereafter, we choose and fix an arbitrary $(x,t) \in B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}} - 6\sqrt[4]{t_{i+2}}) \times (t_{i+1}, t_{i+2}]$. Let $r = \sqrt[4]{t_{i+2}}$ and $R = 3r$, then by triangle inequality, $B_{g(0)}(x, R+2r) \subset\subset B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}} - \sqrt[4]{t_{i+2}})$, where by Claim 7.1 we have $\mathrm{Ric}_{g(s)} \geq -K(v_0,n)$ for all $s \in [0, t_{i+2}]$. We now apply Lemma 4.1 to the flow on $B_{g(0)}(x, R+2r)$ during $[0, t_{i+2}]$, and obtain a cut-off function $\phi_{i+1}$ such that

$$B_{g(s)}(x,r) \subset B_{g(0)}(x,2r) \subset \{y \,|\, \phi_{i+1}(y,s) = 1\} \tag{7.12}$$

and $supp\,\phi_{i+1}(\cdot,s) \subset B_{g(0)}(x,3r)$, for all $s \in [0, t_{i+2}]$. Combining with (7.12), we find that the supports of $|\nabla\phi_{i+1}|$, $\frac{\partial}{\partial s}\phi_{i+1}$ and $\Delta\phi_{i+1}$ are all contained in the annulus $A_{2r,3r}(x) := B_{g(0)}(x,3r) - B_{g(0)}(x,2r)$ and we have the following estimates:

**(P1)** $\nabla\phi_{i+1}$ exists a.e. and $|\nabla\phi_{i+1}| \leq= C\,t_{i+2}^{-\frac{n+1}{4}}$;
**(P2)** $\Delta\phi_{i+1} \leq \mu_1 := C\,t_{i+2}^{-\frac{n+1}{2}}$, in the barrier sense;
**(P3)** $\frac{\partial^+}{\partial s}\phi_{i+1} \leq \mu_2 := C\,t_{i+2}^{-\frac{n}{4}}$.

In view of (7.12) we have $\phi_{i+1}(x,t) = 1$ and hence

$$\mathcal{L}(x,t) = \lim_{s\nearrow t}\int G(x,t;y,s)\mathcal{L}(y,s)\phi_{i+1}(y,s)d_sy. \tag{7.13}$$

The integration domain here and below is always $B_{g(0)}(x,3r)$. In particular, for any integral involving $\nabla\phi_{i+1}$, $\frac{\partial}{\partial s}\phi_{i+1}$ or $\Delta\phi_{i+1}$, the actual integration domain is contained in $A_{2r,3r}(x)$ since these derivatives vanish at the outside.

Since $G(x,t;\cdot,\cdot)$ is continuous on $B_{g(0)}(x,3r)\times[0,t)$ and smooth on $B_{g(0)}(x,3r)\times(t_j,\min(t_{j+1},t))$ for each $j\le i+1$, applying Lemma 2.11 to $G(x,t;y,s)\phi_{i+1}(y,s)$ and $\mathcal{L}(y,s)$ and using (P1)–(P3) we obtain

$$\int G\,\phi_{i+1}\mathcal{L}\Big|_{t_j}^{\min(t_{j+1},t)} \le \int_{t_j}^{\min(t_{j+1},t)}\int (G\,\mu_1+G\,\mu_2+2\,\langle\nabla G,\nabla\phi_{i+1}\rangle)\mathcal{L}, \tag{7.14}$$

and hence

$$\mathcal{L}(x,t)\le \int G\,\phi_{i+1}\mathcal{L}\Big|_{t_1}+\int_{t_1}^{t}\int (G\,\mu_1+G\,\mu_2+2\,\langle\nabla G,\nabla\phi_{i+1}\rangle)\mathcal{L}. \tag{7.15}$$

To estimate the first term in the RHS of (7.15), we first note that on $B_{g(0)}(x,3r)\subset B_{g(0)}(x_0,r_i-4\sqrt{\frac{t_{i+1}}{\tau}}-\sqrt[4]{t_{i+2}})$ we have $\mathcal{L}(\cdot,t_1)\le 2\alpha_0<1$ and hence $\mathrm{Ric}_{g(t_1)}\ge -C(n)$ for some dimensional constant $C(n)$. Then applying Lemma 2.7 we get

$$\int G\,\phi_{i+1}\,\mathcal{L}\Big|_{t_1}\le C\cdot 2\alpha_0. \tag{7.16}$$

Then we split the second term in the RHS of (7.15) into two parts:

$$\mathcal{I}=\int_{t_1}^{t}\int (\mu_1+\mu_2)\,G(x,t;y,s)\,\mathcal{L}(y,s)\,d_sy\,ds, \tag{7.17}$$

$$\mathcal{J}=2\int_{t_1}^{t}\int \langle\nabla G(x,t;y,s),\nabla\phi_{i+1}(y,s)\rangle\,\mathcal{L}(y,s)\,d_sy\,ds. \tag{7.18}$$

On the one hand, by the Shrinking Lemma, for all $y$ in $A_{2r,3r}(x)$ and $s\in[0,t_{i+2}]$, we have $d_{g(s)}(x,y)\ge d_{g(0)}(x,y)-\sqrt[4]{t_{i+2}}\ge\sqrt[4]{t_{i+2}}$. Thus by Lemma 2.9 we have

$$G(x,t;y,s)\le\frac{C}{(t-s)^{\frac{n}{2}}}exp\left(-\frac{\sqrt{t_{i+2}}}{C(t-s)}\right)\le\frac{C}{t_{i+2}^{\frac{n}{2}}}exp\left(-\frac{1}{C\sqrt{t_{i+2}}}\right). \tag{7.19}$$

On the other hand, let $V(s)$ be the volume of $A_{2r,3r}(x)$ at time $s\in[0,t_{i+2}]$. By Bishop–Gromov comparison, we have $V(0)\le C(n)$. Then we get $V(s)\le C$ by integrating $V'(s)\le C\,V(s)$, which follows from the evolution equation of volume under Ricci flow and Claim 7.1. Combining this with (7.19), (P2), (P3) and Claim 7.1 in (7.17) we can estimate

$$\mathcal{I} \leq C\, exp\left(-\frac{1}{C\sqrt{t_{i+2}}}\right). \tag{7.20}$$

Suppose $s \in (t_j, \min(t_{j+1}, t))$ for some $j \leq i+1$. Since $d_{g(s)}(x,y) \geq \sqrt{t_{i+2}}$ for all $y \in A_{2r,3r}(x)$, applying Claim 5.8 of the estimate of $|\nabla G|$, we obtain

$$|\nabla G|(x,t;y,s) \leq \frac{C}{\sqrt{s-t_j}} exp(-\frac{1}{C\sqrt{t_{i+2}}}), \tag{7.21}$$

where the constant $C$ depending on $n$ and $v_0$ is uniform for all $j$. Then by Claim 7.1 and (P1) we have

$$|\langle \nabla G, \nabla \phi_{i+1}\rangle|(y,s)\mathcal{L}(y,s) \leq \frac{C}{\sqrt{s-t_j}} exp(-\frac{1}{C\sqrt{t_{i+2}}}). \tag{7.22}$$

Integrating (7.22) over $A_{2r,3r}(x) \times [t_j, t_{j+1}]$, and then summing over all $j$, we obtain

$$\begin{aligned}
\mathcal{J} &\leq Cexp(-\frac{1}{C\sqrt{t_{i+2}}})\sum_{j=1}^{i+1}\sqrt{t_{j+1}-t_j} \\
&= Cexp(-\frac{1}{C\sqrt{t_{i+2}}})\sqrt{t_{i+2}(1-\frac{1}{\nu})}(1+\frac{1}{\sqrt{\nu}}+\dots) \\
&= Cexp(-\frac{1}{C\sqrt{t_{i+2}}})\frac{\sqrt{(\nu-1)t_{i+2}}}{\sqrt{v}-1} \leq Cexp(-\frac{1}{C\sqrt{t_{i+2}}}).
\end{aligned} \tag{7.23}$$

Putting the estimates (7.16), (7.20) and (7.23) into (7.15), we thus have

$$\mathcal{L}(x,t) \leq Cexp(-\frac{1}{C\sqrt{t_{i+2}}}) + 2\alpha_0 C. \tag{7.24}$$

Then there exists positive constant $t(n, v_0, \alpha_0)$, such that the first term can be bounded by $\alpha_0$ when $t_{i+2} \leq t(n, v_0, \alpha_0)$, and hence $\mathcal{L}(x,t) \leq \alpha_0(1+2C)$. Choose $C_4 = 2(1+2C)$, then $\ell(x,t) \leq \mathcal{L}(x,t)e^{Ct} \leq 2\mathcal{L}(x,t) \leq C_4\alpha_0$. Let $r_{i+1} = r_i - 4\sqrt{\frac{t_{i+1}}{\tau}} - 6\sqrt[4]{t_{i+2}}$, then $\ell(y,s) \leq C_4\alpha_0 < 1$ for any $(y,s) \in B_{g(0)}(x_0, r_{i+1}) \times [0, t_{i+2}]$, as claimed in (APA 3). Moreover, by choosing $t(n, v_0, \alpha_0)$ small, we can make sure

$$r_0 - r_{i+1} = \sum_{j=0}^{i} r_j - r_{j+1} \leq \sum_{j=0}^{i} 4\sqrt{\frac{t_{j+1}}{\tau}} + 6\sqrt[4]{t_{j+2}} \leq 1. \tag{7.25}$$

So we proved Theorem 1.1.

## 8. Proof of the global existence and bi-Hölder homeomorphism

In this section we prove Corollary 1.2 and 1.3. We need two local curvature estimate lemmas stated below, in both of which the Riemannian manifolds $(M^n, g)$ appearing are not necessarily complete.

Lemma 8.1 is proved by B.L. Chen in [6, Theorem 3.1] and Simon in [22, Theorem 1.3].

**Lemma 8.1.** *Suppose $(M^n, g(t))$ is a Ricci flow for $t \in [0, T]$, not necessarily complete, with the property that for some $y_0 \in M$ and $r > 0$, and all $t \in [0, T]$, we have $B_{g(t)}(y_0, r) \subset\subset M$ and*

$$|Rm|_{g(t)} \leq \frac{c_0}{t} \tag{8.1}$$

*on $B_{g(t)}(y_0, r)$ for all $t \in (0, T]$ and some $c_0 \geq 1$. Then if $|Rm|_{g(0)} \leq r^{-2}$ on $B_{g(0)}(y_0, r)$, we must have*

$$|Rm|_{g(t)}(y_0) \leq e^{Cc_0} r^{-2} \tag{8.2}$$

*for some $C = C(n)$.*

Lemma 8.2 is an non-standard version of Shi's derivative estimates. The proof of it can be found in [25, Lemma A.4] and [7, Theorem 14.16].

**Lemma 8.2.** *Suppose $(M^n, g(t))$ is a Ricci flow for $t \in [0, T]$, not necessarily complete, with the property that for some $y_0 \in M$ and $r > 0$, we have $B_{g(0)}(y_0, r) \subset\subset M$ and $|Rm|_{g(t)} \leq r^{-2}$ on $B_{g(0)}(y_0, r)$ for all $t \in [0, T]$, and so that for some $l_0 \in \mathbb{N}$ we have initially $|\nabla^l Rm|_{g(0)} \leq r^{-2-l}$ on $B_{g(0)}(y_0, r)$ for all $l \in \{1, 2, ..., l_0\}$. Then there exists $C = C(l_0, n, \frac{T}{r^2})$ such that*

$$|\nabla^l Rm|_{g(t)}(y_0) \leq C r^{-2-l} \tag{8.3}$$

*for each $l \in \{1, 2, ..., l_0\}$ and all $t \in [0, T]$.*

**Proof of Corollary 1.2.** Pick any point $x_0 \in M$. We apply Theorem 1.1 with $s_0 = k + 2$, for each integer $k \geq 2$, giving a Ricci flow $(B_{g_0}(x_0, k), g_k(t))_{t \in [0,\tau]}$ satisfying

$$\begin{cases} \mathrm{Rm}_{g_k(t)} + C\alpha_0 \mathrm{I} \in \mathcal{C} \\ |\mathrm{Rm}|_{g_k(t)} \leq \dfrac{C}{t} \end{cases} \tag{8.4}$$

on $B_{g_0}(x_0, k)$ for all $t \in (0, \tau]$, where $\tau = \tau(n, v_0, \alpha_0) > 0$ and $C = C(n, v_0) > 0$.

Fix some $r_0 > 0$. Since $B_{g_0}(x_0, r_0 + 2)$ is compactly contained in $M$, we have $\sup |\mathrm{Rm}| \leq \frac{1}{r^2}$ for some $r > 0$, where the supremum is taken over $B_{g_0}(x_0, r_0 + 2)$. For

any $y_0 \in B_{g_0}(x_0, r_0+1)$, by the Shrinking Lemma we have $B_{g_k(t)}(y_0, \frac{1}{2}) \subset B_{g_0}(y_0, 1) \subset B_{g_0}(x_0, r_0+2)$ for all $t \in [0, \tau]$, with possibly reducing $\tau$ to a smaller number depending also on $n, \alpha_0$ and $v_0$. Now we can apply Lemma 8.1 to $g_k(t)$ centered at $y_0$ and get $|\mathrm{Rm}|_{g_k(t)}(y_0) \leq K_0 = K_0(r, n, C)$. In particular, $K_0$ is independent of $k$. Thus, we have $|\mathrm{Rm}|_{g_k(t)} \leq K_0$ on $B_{g_0}(x_0, r_0+1)$ for all $t \in [0, \tau]$.

Then we apply Lemma 8.2 to $g_k(t)$ centered at each $y_0 \in B_{g_0}(x_0, r_0)$. The outcome is for each $l \in \mathbb{N}$, there exists $K_1 = K_1(n, l, r, \tau)$ such that

$$|\nabla^l \mathrm{Rm}|_{g_k(t)} \leq K_1 \tag{8.5}$$

on $B_{g_0}(x_0, r_0)$ for all $t \in [0, \tau]$. Again, the $K_1$ is also independent of $k$. Using these derivative estimates in local coordinate charts, by Ascoli–Arzela Lemma we can pass to a subsequence in $k$ and obtain a smooth limit Ricci flow $g(t)$ on $B_{g_0}(x_0, r_0)$ for $t \in [0, \tau]$ with $g(0) = g_0$, which satisfies

$$\begin{cases} \mathrm{Rm}_{g(t)} + C\alpha_0 \in \mathcal{C} \\ |\mathrm{Rm}|_{g(t)} \leq \dfrac{C}{t} \end{cases} \tag{8.6}$$

on $B_{g_0}(x_0, r_0)$, for all $t \in (0, \tau]$.

We repeat this process for larger and larger radii $r_i \to \infty$, and take a diagonal subsequence to obtain $g_{k_i}(t)$ which converges on each $B_{g_0}(x_0, r_i)$. Its limit is a smooth Ricci flow $g(t)$ on the whole of $M$ for $t \in [0, \tau]$ with $g(0) = g_0$.

By the Shrinking Lemma, $B_{g(t)}(x_0, r) \subset B_{g_0}(x_0, r + \beta\sqrt{Ct}) \subset\subset M$ for all $t \in (0, \tau]$ and $r > 0$. This guarantees that $g(t)$ must be complete for all positive times $t \in (0, \tau]$. □

**Proof of Corollary 1.3.** We apply Corollary 1.2 and Lemma 3.2 to each $(M_i, g_i)$ and obtain a sequence of Ricci flows $(M_i, g_i(t))_{[0,T]}$ with $g_i(0) = g_i$ and $T$ uniform for each $i$, and satisfying the following uniform estimates

$$\begin{cases} \mathrm{Rm}_{g_i(t)} + C\alpha_0 \mathrm{I} \in \mathcal{C} \\ Vol_{g_i(t)} B_{g_i(t)}(x, 1) \geq v > 0 \\ |\mathrm{Rm}|_{g_i(t)} \leq \dfrac{C}{t} \end{cases} \tag{8.7}$$

for all $x \in M_i$ and all $t \in [0, T]$, where constant $C > 0$ depends on $n, v_0$, and constants $v, T > 0$ depend on $n, v_0, \alpha_0$. And

$$d_{g_i(t_1)}(x, y) - \beta\sqrt{C}(\sqrt{t_2} - \sqrt{t_1}) \leq d_{g_i(t_2)}(x, y) \leq e^{K(t_2 - t_1)} d_{g_i(t_1)}(x, y) \tag{8.8}$$

for any $0 < t_1 \leq t_2 \leq T$ and any $x, y \in M_i$, where $K$ depends on $n, v_0, \alpha_0$. The curvature decay for all positive times provides a uniform bound on the curvature which allows us to apply Hamilton's compactness theorem. We can pass to a subsequence in $i$ so

that $(M_i, g_i(t), x_i) \to (M, g(t), x_\infty)$ in the Cheeger–Gromov sense, where $(M, g(t))$ is a complete Ricci flow defined over $(0, T]$. $(M, g(t))$ inherits the estimates for curvatures and distance:

$$\begin{cases} \mathrm{Rm}_{g(t)} + C\alpha_0 \mathrm{I} \in \mathcal{C} \\ Vol_{g(t)} B_{g(t)}(x, 1) \geq v > 0 \\ |\mathrm{Rm}|_{g(t)} \leq \dfrac{C}{t} \end{cases} \tag{8.9}$$

and

$$d_{g(t_1)}(x, y) - \beta\sqrt{C}(\sqrt{t_2} - \sqrt{t_1}) \leq d_{g(t_2)}(x, y) \leq e^{K(t_2 - t_1)} d_{g(t_1)}(x, y) \tag{8.10}$$

for any $0 < t_1 \leq t_2 \leq T$, and any $x, y \in M$. The inequality (8.10) tells us that $d_{g(t)}$ converges locally uniformly to some metric $d_0$ as $t \searrow 0$. Also, the inequality (8.8) tells us that $d_{g_i(t)}$ converges locally uniformly to $d_{g_i}$. So we have $(M_i, d_{g_i}, x_i) \to (M, d_0, x_\infty)$ in the pointed Gromov–Hausdorff sense.

By Lemma 2.3 we have

$$\gamma d_0(x, y)^{1+2(n-1)C} \leq d_{g(t)}(x, y) \leq e^{Kt} d_0(x, y), \tag{8.11}$$

where $\gamma$ depends on $n, v_0$ and also on $\alpha_0$ via $T$. Then the claim of bi-Hölder homeomorphism follows immediately from this. □

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