KP Solitons from Tropical Limits

joint with Agostini D., Fevola C., and Sturmfels B.

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The Kadomtsev-Petviashvili equation

The KP equation is a PDE that describes the motion of water waves

$$\frac{\partial}{\partial x}\left(4p_t - 6pp_x - p_{xxx}\right) = 3p_{yy}$$

where $p = p(x, y, t)$

Taken in Nuevo Vallarta, Mexico by Mark J. Ablowitz

Connection to Algebraic Curves

We seek solutions of the form

$$p(x,y,t) \;=\; 2\frac{\partial^2}{\partial x^2}\log\tau(x,y,t)$$

where $\tau(x,y,t)$ satisfies the Hirota's differential equation

$$\tau\tau_{xxxx} - 4\tau_{xxx}\tau_x + 3\tau_{xx}^2 \;+\; 4\tau_x\tau_t - 4\tau\tau_{xt} + 3\tau\tau_{yy} - 3\tau_y^2 \;=\; 0$$

- One can construct τ-functions from an algebraic curve C of genus g

Connection to Algebraic Curves

Definition

The Riemann theta function is the complex analytic function

$$\theta = \theta(\mathbf{z}|B) = \sum_{\mathbf{c}\in\mathbb{Z}^g} \exp\left[\frac{1}{2}\mathbf{c}^T B\mathbf{c} + \mathbf{c}^T\mathbf{z}\right]$$

where $\mathbf{z}\in\mathbb{C}^g$ and B is a Riemann matrix, a $g\times g$ symmetric matrix normalized to have negative definite real part.

Connection to Algebraic Curves

In 1997, Krichever proved that the KP equation has solutions of the form

$$p(x, y, t) \ = \ 2\frac{\partial^2}{\partial x^2}\log\theta(\mathbf{u}x + \mathbf{v}y + \mathbf{w}t, B)$$

for certain vectors $\mathbf{u} = (u_1, \ldots, u_g), \mathbf{v} = (v_1, \ldots, v_g), \mathbf{w} = (w_1, \ldots, w_g) \in \mathbb{C}^g$.

Connection to Algebraic Curves

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$$p(x,y,t) \;=\; 2\frac{\partial^2}{\partial x^2}\log\theta(\mathbf{u}x+\mathbf{v}y+\mathbf{w}t,B)$$

for certain vectors $\mathbf{u}=(u_1,\ldots,u_g),\mathbf{v}=(v_1,\ldots,v_g),\mathbf{w}=(w_1,\ldots,w_g)\in\mathbb{C}^g$.

Now, for a specific curve C of genus g with Riemann matrix B, we can look for τ of the form

$$\tau(x,y,t)=\theta(\mathbf{u}x+\mathbf{v}y+\mathbf{w}t,B).$$

Connection to Algebraic Curves

Consider $(u_1, \ldots, u_g, v_1, \ldots, v_g, w_1, \ldots, w_g)$ as a point in $\mathbb{WP}^{3g-1}$ such that

$$\deg(u_i) = 1\,, \qquad \deg(v_i) = 2\,, \qquad \deg(w_i) = 3 \qquad \text{for } i = 1, 2, \ldots, g$$

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Definition (Agostini-Çelik-Sturmfels, 2020)

The Dubrovin threefold $\mathscr{D}_C$ comprises all points $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ in $\mathbb{WP}^{3g-1}$ such that $\tau(x, y, t)$ satisfies the Hirota's differential equation.

Soliton Solutions

Fix $k < n$ and a vector of parameters $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{R}^n$ and consider

$$\tau(x,y,t) \;=\; \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{\substack{i,j \in I \\ i<j}} (\kappa_j - \kappa_i) \cdot \exp\left[\, x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \,\right]$$

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Proposition (Sato)

The function τ is a solution to Hirota's differential equation if and only if the point $p = (p_I)_{I \in \binom{[n]}{k}}$ lies in the Grassmannian $\mathrm{Gr}(k,n)$.

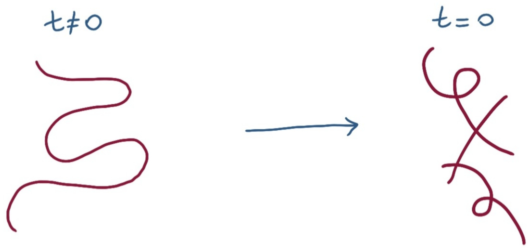
Definition

We define a (k,n)-soliton to be any function $\tau(x,y,t)$ where $\kappa \in \mathbb{R}^n$ and $p \in \mathrm{Gr}(k,n)$.

Main Idea

We study solutions to the KP equations arising from algebraic curves defined over a non-archimedean field $\mathbb{K}$, like $\mathbb{Q}(\epsilon)$ or $\mathbb{C}\{\{\epsilon\}\}$.

A curve over $\mathbb{K}$ can be thought of as a family of curves depending on a parameter ϵ



$$\theta(\mathbf{z}) = \sum_{\mathbf{c}\in\mathbb{Z}^g} \exp\left[\frac{1}{2}\mathbf{c}^T B\mathbf{c} + \mathbf{c}^T\mathbf{z}\right] \quad \rightsquigarrow \quad \theta_{\mathscr{C}}(\mathbf{x}) = \sum_{\mathbf{c}\in\mathscr{C}} a_{\mathbf{c}} \exp[\mathbf{c}^T\mathbf{z}]$$

Main Idea

We study solutions to the KP equations arising from algebraic curves defined over a non-archimedean field $\mathbb{K}$, like $\mathbb{Q}(\epsilon)$ or $\mathbb{C}\{\{\epsilon\}\}$.

For $\epsilon \to 0$

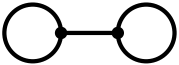
- The theta function becomes a finite sum of exponentials
- The function

$$p(x, y, t) = 2\frac{\delta^2}{\delta x^2}\log\tau(x, y, t)$$

  becomes a soliton solution of the KP equation

Degenerations of Theta Functions

Let X be a smooth curve of genus g over $\mathbb{K}$. The metric graph is $\mathrm{Trop}(X)$.



The metric graph $\Gamma = (V, E)$ of a genus 2 hyperelliptic curve

$$H_1(\Gamma, \mathbb{Z}) = \langle \gamma_1, \ldots, \gamma_g \rangle$$

is a free abelian group of rank g

- $e := |E|$
- $\Lambda := g \times e$ matrix whose i-th row records the coordinate of γ_i with respect to the standard basis of $\mathbb{Z}^e$
- $\Delta :=$ diagonal $e \times e$ matrix that records edge lengths of the metric graph.

Definition

The Riemann matrix of $\Gamma = (V, E)$ is

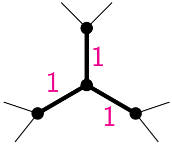
$$Q = \Lambda \Delta \Lambda^T$$

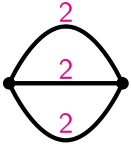
Example (g=2)

Consider $X := \{ y^2 = f(x) \}$ where

$$f(x) = (x-1)(x-1-\epsilon)(x-2)(x-2-\epsilon)(x-3)(x-3-\epsilon)$$

The six roots determine a subtree with six leaves which has a unique hyperelliptic covering by a metric graph of genus 2





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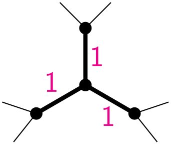

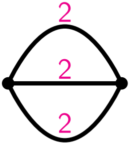

From the graph we can read off the tropical Riemann matrix Q

$$Q = \Lambda\Delta\Lambda^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$$

Degenerations of Theta Functions

Consider

$$B_\epsilon = -\frac{1}{\epsilon} Q + R(\epsilon)$$

Fix $\mathbf{a} \in \mathbb{R}^g$

$$\theta(\mathbf{z} + \frac{1}{\epsilon} Q\mathbf{a} \,|\, B_\epsilon) \;=\; \sum_{\mathbf{c} \in \mathbb{Z}^g} \exp\left[-\frac{1}{2\epsilon}\mathbf{c}^T Q\mathbf{c} + \frac{1}{\epsilon}\mathbf{c}^T Q\mathbf{a}\right] \cdot \exp\left[\frac{1}{2}\mathbf{c}^T R(\epsilon)\mathbf{c} + \mathbf{c}^T\mathbf{z}\right]$$

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Let $\epsilon \to 0$. This converges provided

$$\mathbf{c}^T Q\mathbf{c} - 2\mathbf{c}^T Q\mathbf{a} \geq 0 \text{ for all } \mathbf{c} \in \mathbb{Z}^g$$

or equivalently

$$\mathbf{a}^T Q\mathbf{a} \leq (\mathbf{a} - \mathbf{c})^T Q(\mathbf{a} - \mathbf{c}) \quad \text{for all } \mathbf{c} \in \mathbb{Z}^g$$

Voronoi and Delaunay

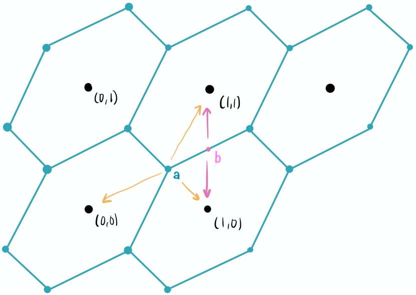
The condition

$$\mathbf{a}^T Q\mathbf{a} \leq (\mathbf{a}-\mathbf{c})^T Q(\mathbf{a}-\mathbf{c}) \quad \text{for all } \mathbf{c} \in \mathbb{Z}^g$$

holds if and only if $\mathbf{a}$ belongs to the *Voronoi cell* for Q

For $\mathbf{a}$ in the Voronoi cell for Q,
consider the associated Delaunay set:

$$\mathscr{D}_{\mathbf{a},Q} = \left\{\mathbf{c} \in \mathbb{Z}^g : \mathbf{a}^T Q\mathbf{a} = (\mathbf{a}-\mathbf{c})^T Q(\mathbf{a}-\mathbf{c})\right\}$$



$$\theta(\mathbf{z}+\frac{1}{\epsilon}Q\mathbf{a}\,|\,B_\epsilon) \;=\; \sum_{\mathbf{c}\in\mathbb{Z}^g}\exp\left[-\frac{1}{2\epsilon}\mathbf{c}^TQ\mathbf{c}+\frac{1}{\epsilon}\mathbf{c}^TQ\mathbf{a}\right]\cdot\exp\left[\frac{1}{2}\mathbf{c}^TR(\epsilon)\mathbf{c}+\mathbf{c}^T\mathbf{z}\right]$$

Theorem (Agostini-Fevola-M.-Sturmfels, 2021)

Fix $\mathbf{a}$ in the Voronoi cell of the tropical Riemann matrix Q. For $\epsilon\to 0$, the series

$$\theta(\mathbf{z}+\frac{1}{\epsilon}Q\mathbf{a}\,|\,B_\epsilon)$$

converges to the theta function supported on the Delaunay set $\mathscr{C}=\mathscr{D}_{\mathbf{a},Q}$, namely

$$\theta_{\mathscr{C}}(\mathbf{x}) \;=\; \sum_{\mathbf{c}\in\mathscr{C}} a_{\mathbf{c}}\exp\left[\mathbf{c}^T\mathbf{z}\right], \quad \text{where} \quad a_{\mathbf{c}}=\exp\left[\frac{1}{2}\mathbf{c}^TR(0)\mathbf{c}\right]$$

Example (g=2)

Theorem (Agostini-Fevola-M.-Sturmfels, 2021)

Fix $\mathbf{a}$ *in the Voronoi cell of* Q *and let* $\mathcal{C} = \mathcal{D}_{\mathbf{a},Q}$ *be the Delaunay set. As* $\epsilon \to 0$,

$$\theta(\mathbf{z} + \frac{1}{\epsilon} Q\mathbf{a} \,|\, B_\epsilon) \to \theta_{\mathcal{C}}(\mathbf{x}) \;=\; \sum_{\mathbf{c}\in\mathcal{C}} a_{\mathbf{c}} \exp[\mathbf{c}^T \mathbf{z}],$$

where $a_{\mathbf{c}} = \exp\left[\frac{1}{2}\mathbf{c}^T R(0)\mathbf{c}\right]$

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where $a_{\mathbf{c}} = \exp\left[\frac{1}{2}\mathbf{c}^T R(0)\mathbf{c}\right]$

Example

For $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\mathcal{C} = \mathcal{D}_{\mathbf{a},Q} = \{(0,0),(1,0),(0,1),(1,1)\}$$

The associated theta function is

$$\theta_{\mathcal{C}} = a_{00} + a_{10}\exp[z_1] + a_{01}\exp[z_2] + a_{11}\exp[z_1+z_2]$$

The Hirota Variety

Let $\mathscr{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\} \subset \mathbb{Z}^g$

$$\theta_{\mathscr{C}}(\mathbf{z}) \;=\; a_1 \exp[\,\mathbf{c}_1^T\mathbf{z}\,] + a_2 \exp[\,\mathbf{c}_2^T\mathbf{z}\,] + \cdots + a_m \exp[\,\mathbf{c}_m^T\mathbf{z}\,]$$

Consider

$$\tau(x,y,t) = \theta_{\mathscr{C}}(\mathbf{u}x+\mathbf{v}y+\mathbf{w}t) = \sum_{i=1}^{m} a_i \exp\Big[\Big(\sum_{j=1}^{g} c_{ij}u_j\Big)\,x + \Big(\sum_{j=1}^{g} c_{ij}v_j\Big)\,y + \Big(\sum_{j=1}^{g} c_{ij}w_j\Big)\,t\Big]$$

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$$\tau(x,y,t) = \theta_{\mathscr{C}}(\mathbf{u}x + \mathbf{v}y + \mathbf{w}t) = \sum_{i=1}^{m} a_i \exp\Big[\big(\sum_{j=1}^{g} c_{ij} u_j\big)\, x + \big(\sum_{j=1}^{g} c_{ij} v_j\big)\, y + \big(\sum_{j=1}^{g} c_{ij} w_j\big)\, t\Big]$$

Definition

The Hirota variety $\mathscr{H}_{\mathscr{C}}$ consists of all points $\big(\mathbf{a}, (\mathbf{u}, \mathbf{v}, \mathbf{w})\big)$ in $(\mathbb{K}^*)^m \times \mathbb{WP}^{3g-1}$ such that $\tau(x,y,t)$ satisfies Hirota's differential equation

Polynomials defining the Hirota Variety

Remark

Hirota's differential equation can be written via the Hirota differential operators as

$$P(D_x, D_y, D_t)\tau \bullet \tau = 0$$

where $P(x, y, t) = x^4 - 4xt + 3y^2$ gives the soliton dispersion relation

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For any two indices k, ℓ in $\{1, \ldots, m\}$

$$P_{k\ell}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \ := \ P\big((\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{u}, \, (\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{v}, \, (\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{w} \big)$$

is a hypersurface in $\mathbb{WP}^{3g-1}$

Polynomials defining the Hirota Variety

The polynomials defining $\mathscr{H}_{\mathscr{C}}$ correspond to points in

$$\mathscr{C}^{[2]} = \{\mathbf{c}_k + \mathbf{c}_\ell \,:\, 1 \le k < \ell \le m\} \subset \mathbb{Z}^g$$

Definition

A point $\mathbf{d}$ in $\mathscr{C}^{[2]}$ is uniquely attained if there exists precisely one index pair (k,ℓ) such that $\mathbf{c}_k + \mathbf{c}_\ell = \mathbf{d}$. In that case, (k,ℓ) is a unique pair.

Polynomials defining the Hirota Variety

Theorem (Agostini-Fevola-M.-Sturmfels)

The Hirota variety $\mathcal{H}_{\mathcal{C}}$ *is defined by the quartics*

$$P_{k\ell}(\mathbf{u},\mathbf{v},\mathbf{w}) \;:=\; P\big(\,(\mathbf{c}_k-\mathbf{c}_\ell)\cdot\mathbf{u},\,(\mathbf{c}_k-\mathbf{c}_\ell)\cdot\mathbf{v},\,(\mathbf{c}_k-\mathbf{c}_\ell)\cdot\mathbf{w}\big)$$

for all unique pairs (k,ℓ) *and by the polynomials*

$$\sum_{\substack{1\le k<\ell\le m\\ \mathbf{c}_k+\mathbf{c}_\ell=\mathbf{d}}} P_{k\ell}(\mathbf{u},\mathbf{v},\mathbf{w})\,a_k a_\ell$$

for all non-uniquely attained points $\mathbf{d}\in\mathcal{C}^{[2]}$*.*
If all points in $\mathcal{C}^{[2]}$ *are uniquely attained then* $\mathcal{H}_{\mathcal{C}}$ *is defined by the* $\binom{m}{2}$ *quartics* $P_{k\ell}(\mathbf{u},\mathbf{v},\mathbf{w})$*.*

Example (The Square)

Let $g = 2$ and $\mathscr{C} = \{0,1\}^2$

$$\mathscr{C}^{[2]} = \{(0,1),(1,0),(1,1),(1,2),(2,1)\}$$

There are four unique pairs (k,ℓ)

$$P_{13} = P_{24} = u_1^4 - 4u_1 w_1 + 3v_1^2$$

$$P_{12} = P_{34} = u_2^4 - 4u_2 w_2 + 3v_2^2$$

The point $\mathbf{d} = (1,1)$ is not uniquely attained in $\mathscr{C}^{[2]}$

$$P(u_1+u_2, v_1+v_2, w_1+w_2)\, a_{00} a_{11} \;+\; P(u_1-u_2, v_1-v_2, w_1-w_2)\, a_{01} a_{10}$$

For any point in $\mathscr{H}_{\mathscr{C}} \subset (\mathbb{K}^*)^4 \times \mathbb{WP}^5$, we can write $\tau(x,y,t)$ as a $(2,4)$-soliton

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Example (The Cube)

Let $g = 3$ and consider the tropical degeneration of a smooth plane quartic C to a rational quartic

$$\begin{aligned}\theta_{\mathscr{C}} \;=\; & a_{000} + a_{100}\exp[z_1] + a_{010}\exp[z_2] + a_{001}\exp[z_3] + a_{110}\exp[z_1 + z_2] \\ & + a_{101}\exp[z_1 + z_3] + a_{011}\exp[z_2 + z_3] + a_{111}\exp[z_1 + z_2 + z_3].\end{aligned}$$

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We compute the Hirota variety in $(\mathbb{K}^*)^8 \times \mathbb{WP}^8$.
The set $\mathscr{C}^{[2]}$ consists of 19 points.

- 12 pts uniquely attained, one for each edge of the cube → quartics $u_j^4 - 4u_jw_j + 3v_j^2$, one for each edge direction $\mathbf{c}_k - \mathbf{c}_\ell$
- 6 pts attained twice → contribute 6 equations, one for each facet
- $(1,1,1)$ four times $\to P(u_1+u_2+u_3,\, v_1+v_2+v_3,\, w_1+w_2+w_3)\, a_{000}a_{111}$

$$\begin{aligned}&+P(u_1+u_2-u_3,\, v_1+v_2-v_3,\, w_1+w_2-w_3)\, a_{001}a_{110}\\ &+P(u_1-u_2+u_3,\, v_1-v_2+v_3,\, w_1-w_2+w_3)\, a_{010}a_{101}\\ &+P(-u_1+u_2+u_3,\, -v_1+v_2+v_3,\, -w_1+w_2+w_3)\, a_{100}a_{011}\end{aligned}$$

The Sato Grassmannian: Some Intuition

Classical Grassmannian $\mathrm{Gr}(k,n)$

- parametrizes k-dimensional subspaces of $\mathbb{K}^n$
- projective variety in $\mathbb{P}^{\binom{n}{k}-1}$, cut out by quadratic relations known as *Plücker relations*
- Plücker coordinates $p_I = c_\lambda$ are indexed by k-element subsets of $[n]$, identified with partitions λ that fit into a $k \times (n-k)$ rectangle.These are the maximal minors of a $k \times n$ matrix M of unknowns

The *Sato Grassmannian* (SGM)

- the zero set of the Plücker relations in the unknowns c_λ, where we now drop the constraint that λ fits into a $k \times (n-k)$-rectangle.
- we now allow arbitrary partitions λ

The SGM is a device for encoding all solutions to the KP equation!

The Sato Grassmannian

Let $V = \mathbb{K}((z))$ and consider the natural projection map

$$\pi \colon V \to \mathbb{K}[z^{-1}]$$

Definition

Points in the Sato Grassmannian SGM correspond to $\mathbb{K}$-subspaces $U \subset V$ such that

$$\dim \operatorname{Ker} \pi_{|U} = \dim \operatorname{Coker} \pi_{|U} < \infty$$

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How do we represent a point in SGM?

For any basis $(f_1, f_2, f_3, \dots)$ of $U \in \mathrm{SGM}$

$$f_j(z) \; = \; \sum_{i=-\infty}^{+\infty} \xi_{i,j} z^{i+1}.$$

Then U is the column span of the infinite matrix $\xi = (\xi_{i,j})$.

The Sato Grassmannian

A subspace U of V gives a point in SGM if and only if there is a basis whose corresponding matrix has the shape

$$\xi = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & \mathbf{1} & 0 & 0 & 0 & \cdots & 0 \\ \cdots & * & \mathbf{1} & 0 & 0 & \cdots & 0 \\ \cdots & * & * & \xi_{-\ell,\ell} & \xi_{-\ell,\ell-1} & \cdots & \xi_{-\ell,1} \\ \cdots & * & * & \xi_{-\ell+1,\ell} & \xi_{-\ell+1,\ell-1} & \cdots & \xi_{-\ell+1,1} \\ & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \cdots & * & * & \xi_{-1,\ell} & \xi_{-1,\ell-1} & \cdots & \xi_{-1,1} \\ \cdots & * & * & \xi_{0,\ell} & \xi_{0,\ell-1} & \cdots & \xi_{0,1} \\ \cdots & * & * & \xi_{1,\ell} & \xi_{1,\ell-1} & \cdots & \xi_{1,1} \\ & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \end{pmatrix}.$$

Connections of the SGM to solutions to the KP equation

Definition

A Maya diagram is an infinite sequence $M = (m_1, m_2, m_3, \dots)$ of integers $m_1 > m_2 > m_3 > \dots$ such that $m_i = -i$ for all i large enough.

- Maya diagrams correspond to partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$
- It makes sense to define

$$\xi_\lambda := \xi_M = \det(\xi_{m_i,j})$$

The τ-function associated to ξ is

$$\tau(t_1, t_2, t_3, \dots) = \sum_\lambda \xi_\lambda \sigma_\lambda(t_1, t_2, t_3, \dots)$$

Connections of the SGM to solutions to the KP equation

Definition

A Maya diagram is an infinite sequence $M = (m_1, m_2, m_3, \dots)$ of integers $m_1 > m_2 > m_3 > \dots$ such that $m_i = -i$ for all i large enough.

- Maya diagrams correspond to partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$
- It makes sense to define

$$\xi_\lambda := \xi_M = \det(\xi_{m_i,j})$$

The τ-function associated to ξ is

$$\tau(t_1, t_2, t_3, \dots) = \sum_\lambda \xi_\lambda \sigma_\lambda(t_1, t_2, t_3, \dots)$$

Theorem (Sato)

The function $\tau(\mathbf{t}; \xi)$ is a solution to the KP hierarchy. Conversely, any solution of the KP hierarchy is of this form.

Soliton Solutions & Sato's Solutions

Recall

$$\tau(x,y,t) = \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{\substack{i,j \in I \\ i<j}} (\kappa_j - \kappa_i) \cdot \exp\left[x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \right]$$

and

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and

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Proposition (Agostini-Fevola-M.-Sturmfels)

The (k,n)-soliton has the following expansion into Schur polynomials

$$\tau(x,y,t) \;=\; \sum_{\lambda_1\geq\lambda_2\geq\lambda_3\geq 0} c_\lambda\cdot\sigma_\lambda(x,y,t)\,, \qquad \text{where} \quad c_\lambda = \sum_{I\in\binom{[n]}{k}} p_I\cdot\Delta_\lambda(\kappa_i, i\in I)$$

Tau Functions from Algebraic Curves

Let X be a smooth projective curve of genus g defined over a field $\mathbb{K}$.
Fix a divisor D of degree $g-1$ on X and a point $p \in X$

For $m < n$

$$H^0(X, D+mp) \subseteq H^0(X, D+np) \subseteq \cdots \subseteq H^0(X, D+\infty p)$$

Let z denote a local coordinate on X at p and $m = \mathrm{ord}_p(D)$

$$\iota:\ H^0(X, D+\infty p) \to V, \qquad s = \sum_{n\in\mathbb{Z}} s_n z^n \mapsto \sum_{n\in\mathbb{Z}} s_n z^{n+m+1}$$

Proposition (Segal-Wilson)

The space $U = \iota(H^0(X, D+\infty p)) \subset V$ lies in SGM.

The case of genus 2 hyperelliptic curves

Let

$$X = \left\{ y^2 = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_6) \right\}$$

Let p be one of the two preimages of p_∞ under the double cover $X \to \mathbb{P}^1$.
Let $z = \frac{1}{x}$ around p

$$y = \pm\sqrt{(x-\lambda_1)\cdots(x-\lambda_6)} = \pm\frac{1}{z^3} \cdot \sum_{n=0}^{+\infty} \alpha_n z^n$$

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We consider three kinds of divisors:

$$D_0 = p, \qquad D_1 = p_1, \qquad D_2 = p_1 + p_2 - p$$

For $m \geq 3$, consider the functions

$$\begin{aligned} g_m(x) &= \textstyle\sum_{j=0}^{m}\alpha_j x^{m-j} \\ f_m(x,y) &= \tfrac{1}{2}\left(x^{m-3}y + g_m(x)\right) \\ h_j(x,y) &= \tfrac{f_3(x,y)-f_3(c_j,-y_j)}{x-c_j} = \tfrac{y+g_3(x)-(-y_j+g_3(c_j))}{2(x-c_j)} \quad \text{for } j = 1,2 \end{aligned}$$

We write $U_i = \iota(H^0(X, D_i + \infty p))$

Proposition (Nakayashiki)

The set $\{1, f_3, f_4, f_5, \ldots\}$ *is a basis of* U_0*, the set* $\{1, f_3, f_4, f_5, \ldots\} \cup \{h_1\}$ *is a basis of* U_1*, and* $\{1, f_3, f_4, f_5, \ldots\} \cup \{h_1, h_2\}$ *is a basis of* U_2*.*

Computations

We implemented the method in Maple for $D_0 = p$ on hyperelliptic curves over $\mathbb{K} = \mathbb{Q}(\epsilon)$

$$\tau[n] := \sum_{i=1}^{n} \sum_{\lambda \vdash i} \xi_\lambda \sigma_\lambda(x, y, t),$$

You can find it here:

$$https://mathrepo.mis.mpg.de$$

Thank you!