Non-Rigid Rank-One Infinite Measures on the Circle
Joint Mathematics Meetings

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$S_1$ is unit circle. We also use $m$ for Lebesgue measure on circle, since you can think of $S_1$ as $[0,1)$ mod 1. 

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Introduction

What can happen in a measurable dynamical system?

- Super boring example: $T$ is just the identity. So nothing happens: all sets are invariant, $TA = A$ a.e..

More interesting: "No" proper subsets are invariant under $T$. This is what is called Ergodic. For all $n \neq 0$, $T^n$ is ergodic. This is called totally ergodic.

When we write $B = A \mod m$, we mean $m(B \setminus A) + m(A \setminus B) = 0$.

Definition

A transformation $T$ is ergodic if for every measurable $A$, $TA = A \mod m$ if and only if $m(A) = 0$ or $m(A^c) = 0$. 
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Eigenvalues of $T$

**Definition**

Let $T$ be a measure preserving transformation. A number $\lambda \in \mathbb{C}$ is an $L^\infty$ eigenvalue of $T$ if there exists a nonzero a.e. function $f \in L^\infty$ such that

$$f \circ T = \lambda f \text{ a.e.}.$$ 

$f$ is called an eigenfunction of $T$.
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**Definition**

An ergodic transformation $T$ is called **weakly mixing** if its only eigenvalue is 1.
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- $(\mathbb{S}^1, m, R_\alpha)$ is an example of a dynamical system.
- Rational rotations are boring: not ergodic, $R_{\frac{p}{q}}$ is the identity.
Irrational Rotations

- Ergodic.

Totally ergodic: $R_n \alpha = R_n \alpha$. Not weakly mixing: $z_n \circ R \alpha = e^{2\pi i \alpha z_n}$. These are the only eigenvalues.

Most importantly for us... Irrational rotations are rank-one. (Del Junco, 1976)

A goal of our project was to find an infinite-measure rank-one dynamical system with as many of these properties as possible.
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Rank-one transformations are transformations that can be constructed through a method called cutting and stacking a sequence of columns.
Cutting and Stacking

- The space we are working in is \( \mathbb{R} \)
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1. A **column** $C$ of **height** $h$ is composed of $h$ pairwise disjoint intervals stacked on top of one another. These intervals are the **levels** of the column, and they all have the same length.

```
    ____________
    |          |
    |          |
    |          |
    v          v
    h          h
    |          |
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Suppose we have column $C_k$ with height $h_k$. Define $T_k$ to be the transformation that takes each interval to the one on top of it by translation. Notice that $T_k$ is not defined on the top level.
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We show how to create $C_{k+1}$ by cutting and stacking.
2. Cut the column vertically into $b_k$ equally sized subcolumns. In this example $b_k = 3$. 
3. Add desired number of **spacers** on top of each subcolumn. Spacers are just intervals in $\mathbb{R}$ that are not yet included in our column. In this example we put three spacers on the second subcolumn and two spacers on the third subcolumn.
Cutting and Stacking

4. Stack the subcolumns and spacers to form column $C_{k+1}$

$h_k \left\{ \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \right\}

b_k

h_{k+1} = b_k h_k + \# spacers
5. Repeat. This generates an infinite sequence of columns \((C_k)\).
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Let \(X \subset \mathbb{R}\) be the union of the columns. We define our rank-one transformation \(T: X \to X\) by taking the limit of the partially defined transformations \(T_k\). This limit exists since in each successive column, \(T_k\) is defined on more of \(X\), as the measure of the top level decreases. Also, \(T_k = T_j\) on \(C_j\) for all \(k > j\).
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Therefore, a rank-one transformation can be defined just by specifying the number of cuts and spacers for each column.
Properties of Rank-One

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- Generic in the space of measure preserving transformations
- An important class of examples and counterexamples in ergodic theory
Diophantine Approximations

Every irrational number $\alpha$ can be uniquely described as a continued fraction

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}$$
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We can write $\alpha = [a_0; a_1, a_2... ]$, where the $a_k$ are called the coefficients of the continued fraction. The rational numbers

$$\frac{p_k}{q_k} := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + a_{k-1}}}$$

are called the convergents for $\alpha$. The sequence of convergents provides a best Diophantine approximation of $\alpha$ by rational numbers.
Diophantine Approximations

We can express the convergents in terms of the coefficients by the following recursive formulas:

\[ p_k = a_{k-1} p_{k-1} + p_{k-2} \]

\[ q_k = a_{k-1} q_{k-1} + q_{k-2} \]

where \( p_1 = a_0 \) and \( q_1 = 1 \).
The Construction

(Generalizes Del Junco’s 1976 construction)
Given an irrational number $\alpha$ we construct a rank-one transformation $T_\alpha$ that is:

- totally ergodic
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(Generalizes Del Junco’s 1976 construction)
Given an irrational number $\alpha$ we construct a rank-one transformation $T_\alpha$ that is:

- totally ergodic
- not weakly mixing

Based on the coefficients of $\alpha$, this transformation will act either on a finite or infinite measure space $X$. Let $(p_k/q_k)$ be the sequence of convergents of $\alpha$, and $(a_k)$ its coefficients. Let column $C_1$ be the unit interval. Notice that the height of $C_1 = 1 = q_1$. 

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The Construction

(Generalizes Del Junco’s 1976 construction)

Given an irrational number $\alpha$ we construct a rank-one transformation $T_\alpha$ that is:

- totally ergodic
- not weakly mixing
- has $e^{2\pi i \alpha}$ as an eigenvalue
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The Construction

Suppose that column $C_k$ of height $q_k$ has already been constructed. We cut $C_k$ into $a_k$ cuts. We then add $q_{k-1}$ spacers to the rightmost cut. This gives us column $C_{k+1}$ with height $a_k q_k + q_{k-1} = q_{k+1}$ as desired.
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- Numbers of golden type are a countable subset of the irrationals.
- $T_\alpha$ is well defined for all $\alpha$ that are not of golden type.
Measure of $X$

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- This is true for all irrational numbers $\alpha$ except for an uncountable set of measure zero.

Definition

An irrational number $\alpha$ is badly approximable if its coefficients are bounded by some integer $M$. For badly approximable numbers $\sum_{k=1}^{\infty} \frac{1}{M^k} \geq \sum_{k=1}^{\infty} \frac{1}{M^2} = \infty$, so $T_\alpha$ acts on an infinite measure space $X$.

e.g. $p^2 = [1;2,2,2...]$ is an example where $T_\alpha$ acts on a finite measure space $X$.
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- An example where $T_\alpha$ acts on a finite measure space $X$ is $\alpha = [0;1,2,3,4,...] = \frac{I_1(2)}{I_0(2)}$, where $I_1, I_0$ are the modified Bessel functions of the first kind. This is finite, since $\sum_{k=1}^{\infty} \frac{1}{n(n+1)} = 1 < \infty$.
Eigenfunction of $T_\alpha$

We now construct an $L^\infty$ eigenfunction $f$ for $T_\alpha$ with $\lambda = e^{2\pi i \alpha}$ as an eigenvalue.
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Lemma

Define $\epsilon_k := \left| \alpha - \frac{p_k}{q_k} \right|$. Then for all irrational numbers $\alpha$,

$$\sum_{k=1}^{\infty} \epsilon_k q_{k+1} < \infty.$$
Eigenfunction of $T_\alpha$

Let $\lambda = e^{2\pi i \alpha}$

**Definition ($g_k : X \rightarrow S^1$)**

$$g_k(x) = \begin{cases} 
\lambda^n &: \text{for } x \text{ on the } n\text{th level of } C_k \\
0 &: x \in X \setminus C_k 
\end{cases}$$

If $x$ not in last level of $C_k$, $g_k(T_\alpha x) = \lambda g(x)$, so it’s almost an eigenfunction. We want to say this sequence converges to an actual eigenfunction.
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Then the $f_k$ are defined on all of $X$ and are clearly $L^\infty$. 
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We can show

$$\|f_{k+1} - f_k\|_\infty < 2\pi \epsilon_k q_{k+1}.$$
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making $(f_k)$ a Cauchy sequence.
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Finally, we let $f = \lim_{k \to \infty} f_k$.

\[
f(T_\alpha(x)) = \lim_{k \to \infty} f_k(T_\alpha(x)) = \lambda \lim_{k \to \infty} f_k(x) = \lambda f(x).
\]
Let $f$ be the eigenfunction of $T_\alpha$ as defined above.

**Theorem**

$f$ is injective.
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**Proof.**

The sets $\mathcal{G} = \{f(I) : I \text{ is an interval}\}$ generates $f^{-1}_\alpha(\mathcal{B}(S^1))$ and since $f$ is injective, $\mathcal{G}$ separates points. By Blackwell’s theorem, this is sufficient to show that it generates $\mathcal{B}(X)$. 

Eigenfunction of $T_\alpha$

Let $f$ be the eigenfunction of $T_\alpha$ as defined above.

**Theorem**

$f$ is injective.

**Theorem**

$f^{-1}(\mathcal{B}(S^1)) = \mathcal{B}(X)$.

**Proof.**

The sets $\mathcal{G} = \{f(I) : I \text{ is an interval}\}$ generates $f_{\alpha}^{-1}(\mathcal{B}(S^1))$ and since $f$ is injective, $\mathcal{G}$ separates points. By Blackwell’s theorem, this is sufficient to show that it generates $\mathcal{B}(X)$.

$\nu = m \circ f^{-1}$ is a rank-one measure on the circle that is invariant under the rotation!
Invariant measures on $\mathbb{S}^1$.

- The Lebesgue measure $m$ on the circle is also invariant under the rotation
Invariant measures on $\mathbb{S}^1$.

- The Lebesgue measure $m$ on the circle is also invariant under the rotation.

- **Case 1:** If $X$ has finite measure, then $\nu = cm$ for some constant $c$ since an irrational rotation has a unique finite invariant measure up to a constant.
Invariant measures on $S^1$.

- The Lebesgue measure $m$ on the circle is also invariant under the rotation.

**Case 1:** If $X$ has finite measure, then $\nu = cm$ for some constant $c$ since an irrational rotation has a unique finite invariant measure up to a constant.

**Case 2:** If $X$ has infinite measure, then $\nu$ is an infinite measure on the circle that is invariant under rotation. A finite invariant measure for an ergodic transformation cannot have an equivalent infinite invariant measure. In fact, ergodicity gives $\nu \perp m$. 
Isomorphism

Since $f$ is injective and “surjective” ($\nu$ a.e.) onto $\mathbb{S}^1$, it provides an isomorphism between $T_\alpha$ and $R_\alpha$

\[ T_\alpha \]

\[ (X, m) \quad \rightarrow \quad (X, m) \]

\[ f \]

\[ \downarrow \]

\[ (\mathbb{S}^1, \nu) \quad \rightarrow \quad (\mathbb{S}^1, \nu) \]

\[ R_\alpha \]
In the finite case, $T_\alpha$ is isomorphic to the rotation under Lebesgue measure. Therefore, $T_\alpha$ gives an explicit cutting and stacking construction for the rotation. Although it is known that rotations are rank-one, this explicit construction has not been shown before.
Conclusions

Theorem

Suppose $\alpha$ is an irrational number not of golden type. Then we can explicitly construct a rank-one transformation $T_\alpha$ from a subset $X \subset \mathbb{R}$ to itself with the following properties:

1. There is an $L^\infty$ eigenfunction $f_\alpha$ with eigenvalue $e^{2\pi i \alpha}$. Hence, $T_\alpha$ is not weakly mixing.
2. The eigenfunction is injective.
3. $T_\alpha$ is totally ergodic.
4. $T_\alpha$ is infinite-measure preserving for almost every $\alpha$. 

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Open Questions

1. Are the eigenvalues unique?
2. How do these different measures relate to each other?
3. Do they live on fractals?
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