Non-Rigid Rank-One Infinite Measures on the Circle

Joint Mathematics Meetings

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- $\mathbb{S}^1$ is unit circle. We also use m for Lebesgue measure on circle, since you can think of $\mathbb{S}^1$ as $[0,1) \mod 1$.
- A measurable dynamical system is a triple (X, m, T).
- " $\pmod m$" means "up to a null set."

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When we write $B = A \mod m$, we mean

$$m(B \setminus A) + m(A \setminus B) = 0$$

Definition

A transformation T is ergodic if for every measurable A, $TA = A \mod m$ if and only if $m(A) = 0$ or $m(A^c) = 0$.

Eigenvalues of T

Definition

Let T be an measure preserving transformation. A number $\lambda \in \mathbb{C}$ is an L^∞ eigenvalue of T if there exists a nonzero a.e. function $f \in L^\infty$ such that

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Definition

An ergodic transformation T is called weakly mixing if its only eigenvalue is 1.

Rotations

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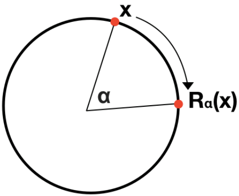
Given any number α, we define the rotation of $\mathbb{S}^1$ by α to be

$$R_\alpha(z) = e^{2\pi i\alpha} z$$



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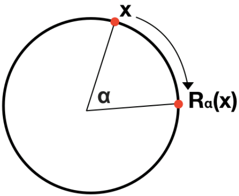

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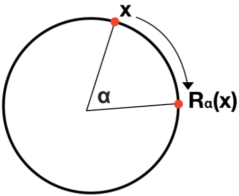

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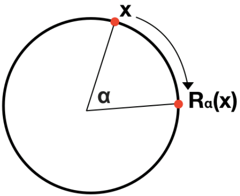

- R_α preserves Lebesgue measure m. This is the unique finite measure that is invariant for irrational rotations (Unique Ergodicity).
- $(\mathbb{S}^1, m, R_\alpha)$ is an example of a dynamical system.
- Rational rotations are boring: not ergodic, $R^q_{p/q}$ is the identity.

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A goal of our project was to find an infinite-measure rank-one dynamical system with as many of these properties as possible.

Rank-one Transformations

Rank-one transformations are transformations that can be constructed through a method called cutting and stacking a sequence of columns.

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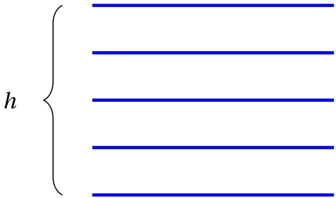
1. A **column** C of **height** h is composed of h pairwise disjoint intervals stacked on top of one another. These intervals are the **levels** of the column, and they all have the same length.



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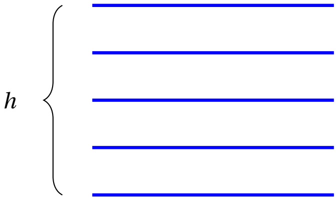

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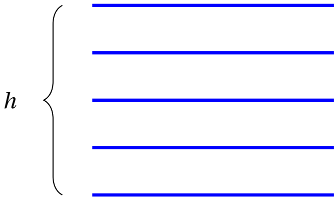

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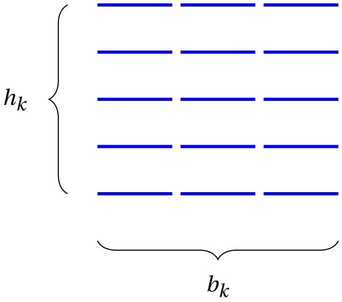
We show how to create C_{k+1} by cutting and stacking.

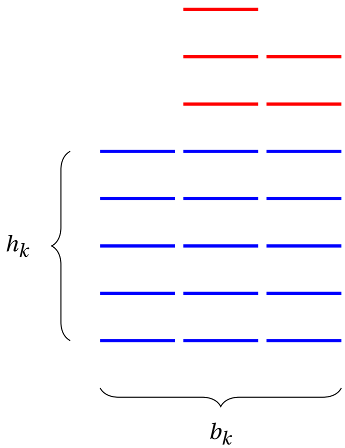
Cutting and Stacking

2. Cut the column vertically into b_k equally sized subcolumns. In this example $b_k = 3$.



Cutting and Stacking

3. Add desired number of spacers on top of each subcolumn. Spacers are just intervals in $\mathbb{R}$ that are not yet included in our column. In this example we put three spacers on the second subcolumn and two spacers on the third subcolumn.



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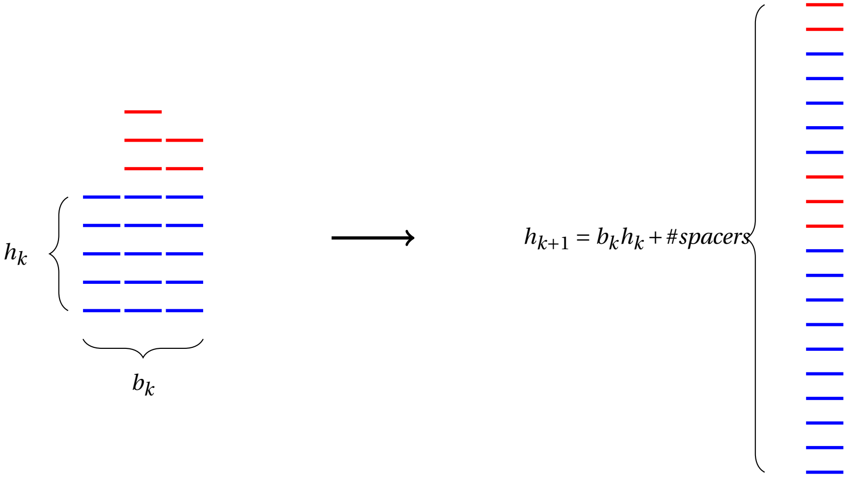
4. Stack the subcolumns and spacers to form column C_{k+1}



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Let $X \subset \mathbb{R}$ be the union of the columns. We define our rank-one transformation $T : X \to X$ by taking the limit of the partially defined transformations T_k. This limit exists since in each successive column, T_k is defined on more of X, as the measure of the top level decreases. Also, $T_k = T_j$ on C_j for all $k > j$.

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Therefore, a rank-one transformation can be defined just by specifying the number of cuts and spacers for each column.

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- Generic in the space of measure preserving transformations
- An important class of examples and counterexamples in ergodic theory

Diophantine Approximations

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We can write $\alpha = [a_0; a_1, a_2 ...]$, where the a_k are called the coe□cients of the continued fraction. The rational numbers

$$\frac{p_k}{q_k} := a_0 + \frac{1}{a_1 + \frac{1}{\ddots + a_{k-1}}}$$

are called the convergents for α. The sequence of convergents provides a best Diophantine approximation of α by rational numbers.

Diophantine Approximations

We can express the convergents in terms of the coefficients by the following recursive formulas:

$$p_k = a_{k-1} p_{k-1} + p_{k-2}$$

$$q_k = a_{k-1} q_{k-1} + q_{k-2}$$

where $p_1 = a_0$ and $q_1 = 1$.

The Construction

(Generalizes Del Junco's 1976 construction)
Given an irrational number α we construct a rank-one transformation T_α that is:

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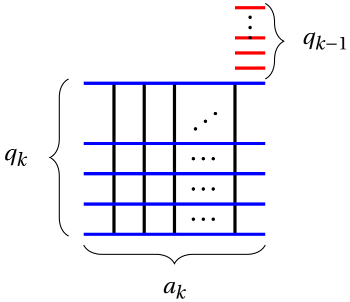
Let $(\frac{p_k}{q_k})$ be the sequence of convergents of α, and (a_k) its coefficients. Let column C_1 be the unit interval. Notice that the height of $C_1 = 1 = q_1$.

The Construction

Suppose that column C_k of height q_k has already been constructed. We cut C_k into a_k cuts. We then add q_{k-1} spacers to the rightmost cut. This gives us column C_{k+1} with height $a_k q_k + q_{k-1} = q_{k+1}$ as desired.



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- Numbers of golden type are a countable subset of the irrationals.
- T_α is well defined for all α that are not of golden type.

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- For badly approximable numbers $\sum_{k=1}^{\infty} \frac{1}{a_k a_{k+1}} \geq \sum_{k=1}^{\infty} \frac{1}{M^2} = \infty$, so T_α acts on an infinite measure space X.
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  - e.g. $\sqrt{2} = [1;2,2,2...]$
- An example where T_α acts on a finite measure space X is $\alpha = [0;1,2,3,4....] = \frac{I_1(2)}{I_0(2)}$, where I_1, I_0 are the modified Bessel functions of the first kind. This is finite, since $\sum_{k=1}^{\infty} \frac{1}{n(n+1)} = 1 < \infty$.

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Lemma

Define $\epsilon_k := \left|\alpha - \frac{p_k}{q_k}\right|$. Then for all irrational numbers α,

$$\sum_{k=1}^{\infty} \epsilon_k q_{k+1} < \infty.$$

Eigenfunction of T_α

Let $\lambda = e^{2\pi i\alpha}$

Definition ($g_k : X \to \mathbb{S}^1$)

$$g_k(x) = \begin{cases} \lambda^n : \text{for } x \text{ on the } n\text{th level of } C_k \\ 0 : x \in X \setminus C_k \end{cases}$$

If x not in last level of C_k, $g_k(T_\alpha x) = \lambda g(x)$, so it's almost an eignenfunction. We want to say this sequence converges to an actual eigenfunction.

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Then the f_k are defined on all of X and are clearly L^∞.

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Finally, we let $f = \lim_{k\to\infty} f_k$.

$$f(T_\alpha(x)) = \lim_{k\to\infty} f_k(T_\alpha(x)) = \lambda \lim_{k\to\infty} f_k(x) = \lambda f(x).$$

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The sets $\mathscr{G} = \{f(I) : I \text{ is an interval}\}$ generates $f_\alpha^{-1}(\mathfrak{B}(S^1))$ and since f is injective, $\mathscr{G}$ separates points. By Blackwell's theorem, this is sufficient to show that it generates $\mathfrak{B}(X)$. □

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$\nu = m \circ f^{-1}$ is a rank-one measure on the circle that is invariant under the rotation!

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- Case 1: If X has finite measure, then $\nu = cm$ for some constant c since an irrational rotation has a unique finite invariant measure up to a constant.

Invariant measures on $\mathbb{S}^1$.

- The Lebesgue measure m on the circle is also invariant under the rotation
- Case 1: If X has finite measure, then $\nu = cm$ for some constant c since an irrational rotation has a unique finite invariant measure up to a constant.
- Case 2: If X has infinite measure, then ν is an infinite measure on the circle that is invariant under rotation. A finite invariant measure for an ergodic transformation cannot have an equivalent infinite invariant measure. In fact, ergodicity gives $\nu \perp m$.

Isomorphism

Since f is injective and "surjective" (ν a.e.) onto $\mathbb{S}^1$, it provides an isomorphism between T_α and R_α

$$\begin{array}{ccc} (X, m) & \xrightarrow{\quad T_\alpha \quad} & (X, m) \\ \Big\downarrow f & & \Big\downarrow f \\ (\mathbb{S}^1, \nu) & \xrightarrow[\quad R_\alpha \quad]{} & (\mathbb{S}^1, \nu) \end{array}$$

Isomorphism

- In the finite case, T_α is isomorphic to the rotation under Lebesgue measure. Therefore, T_α gives an explicit cutting and stacking construction for the rotation. Although it is known that rotations are rank-one, this explicit construction has not been shown before.

Conclusions

Theorem

Suppose α is an irrational number not of golden type. Then we can explicitly construct a rank-one transformation T_α from a subset $X \subset \mathbb{R}$ to itself with the following properties:

1. *There is an L^∞ eigenfunction f_α with eigenvalue $e^{2\pi i\alpha}$. Hence, T_α is not weakly mixing.*
2. *The eigenfunction is injective.*
3. *T_α is totally ergodic.*
4. *T_α is infinite-measure preserving for almost every α.*

Open Questions

1. Are the eigenvalues unique?
2. How do these different measures relate to each other?
3. Do they live on fractals?

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