Hirota Varieties

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The Kadomtsev-Petviashvili equation

The KP equation is a PDE that describes the motion of water waves

$$\frac{\partial}{\partial x}\left(4p_t - 6pp_x - p_{xxx}\right) = 3p_{yy}$$

where $p = p(x, y, t)$

Taken in Nuevo Vallarta, Mexico by Mark J. Ablowitz

Connection to Algebraic Curves

We seek solutions of the form

$$p(x, y, t) = 2\frac{\partial^2}{\partial x^2}\log\tau(x, y, t)$$

where $\tau(x, y, t)$ satisfies the Hirota's differential equation

$$\tau\tau_{xxxx} - 4\tau_{xxx}\tau_x + 3\tau_{xx}^2 + 4\tau_x\tau_t - 4\tau\tau_{xt} + 3\tau\tau_{yy} - 3\tau_y^2 = 0$$

- One can construct τ-functions from an algebraic curve C of genus g

Connection to Algebraic Curves

Definition

The Riemann theta function is the complex analytic function

$$\theta = \theta(\mathbf{z}|B) = \sum_{\mathbf{c}\in\mathbb{Z}^g} \exp\left[\frac{1}{2}\mathbf{c}^T B\mathbf{c} + \mathbf{c}^T\mathbf{z}\right]$$

where $\mathbf{z} \in \mathbb{C}^g$ and B is a Riemann matrix, a $g \times g$ symmetric matrix normalized to have negative definite real part.

Connection to Algebraic Curves

In 1997, Krichever proved that the KP equation has solutions of the form

$$p(x, y, t) \;=\; 2\frac{\partial^2}{\partial x^2}\log\theta(\mathbf{u}x+\mathbf{v}y+\mathbf{w}t, B)$$

for certain vectors $\mathbf{u}=(u_1,\dots,u_g), \mathbf{v}=(v_1,\dots,v_g), \mathbf{w}=(w_1,\dots,w_g) \in \mathbb{C}^g$.

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for certain vectors $\mathbf{u} = (u_1, \ldots, u_g), \mathbf{v} = (v_1, \ldots, v_g), \mathbf{w} = (w_1, \ldots, w_g) \in \mathbb{C}^g$.

Now, for a specific curve C of genus g with Riemann matrix B, we can look for τ of the form

$$\tau(x, y, t) = \theta(\mathbf{u}x + \mathbf{v}y + \mathbf{w}t, B).$$

Connection to Algebraic Curves

Consider $(u_1, \ldots, u_g, v_1, \ldots, v_g, w_1, \ldots, w_g)$ as a point in $\mathbb{WP}^{3g-1}$ such that

$$\deg(u_i) = 1\,, \quad \deg(v_i) = 2\,, \quad \deg(w_i) = 3 \qquad \text{for } i = 1, 2, \ldots, g$$

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Definition (Agostini-Çelik-Sturmfels, 2020)

The Dubrovin threefold $\mathscr{D}_C$ comprises all points $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ in $\mathbb{WP}^{3g-1}$ such that $\tau(x, y, t)$ satisfies the Hirota's differential equation.

The Grassmannian

Definition

Let V be a vector space of dimension n over a field K

$$\mathrm{Gr}(k, V) := \{\, U \subset V \ : \ U \text{ is a } k\text{-dimensional subspace of V} \,\}$$

If $V = K^n$ we write $\mathrm{Gr}(k, V) = \mathrm{Gr}(k, n)$.

Example

- For $k = 1$, $\mathrm{Gr}(1, n) = \mathbb{P}^{n-1}$
- For $k = 2$, $\mathrm{Gr}(2, n)$ is the space of planes through the origin in V

The Grassmannian

The Grassmannian as a manifold

Fix a basis $e_1, \ldots, e_n$ of V. Take $v_1, \ldots, v_k \in V$ linearly independent.

$$W := \langle v_1, \ldots, v_n \rangle \longrightarrow M = \begin{pmatrix} v_{1,1} & v_{1,2} & \ldots & v_{1,n} \\ v_{2,1} & v_{2,2} & \ldots & v_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k,1} & v_{k,2} & \ldots & v_{k,n} \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & \alpha_{1,1} & \ldots & \alpha_{1,n-k} \\ 0 & 1 & \ldots & 0 & \alpha_{2,1} & \ldots & \alpha_{2,n-k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & \alpha_{k,1} & \ldots & \alpha_{k,n-k} \end{pmatrix}$$

The Grassmannian

$$\iota : \{k-\text{dimensional subspaces of } V\} \longrightarrow \mathbb{P}(K^{\binom{n}{k}})$$

Properties:

- The map ι is injective

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- The Grassmannian $\mathrm{Gr}(k,n)$ is a projective variety
- The set $\{p_{i_1,\dots,i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$ is called the Plücker coordinates of the element W of the Grassmannian.

Soliton Solutions

Fix $k < n$ and a vector of parameters $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{R}^n$ and consider

$$\tau(x,y,t) \;=\; \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{\substack{i,j \in I \\ i<j}} (\kappa_j - \kappa_i) \cdot \exp\left[\, x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \,\right]$$

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Proposition (Sato)

The function τ is a solution to Hirota's differential equation if and only if the point $p = (p_I)_{I \in \binom{[n]}{k}}$ lies in the Grassmannian $\mathrm{Gr}(k,n)$.

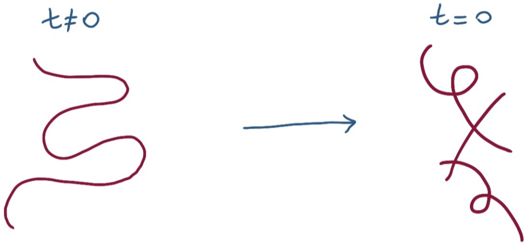
Definition

We define a (k,n)-soliton to be any function $\tau(x,y,t)$ where $\kappa \in \mathbb{R}^n$ and $p \in \mathrm{Gr}(k,n)$.

Main Idea

We study solutions to the KP equations arising from algebraic curves defined over a non-archimedean field $\mathbb{K}$, like $\mathbb{Q}(\epsilon)$ or $\mathbb{C}\{\{\epsilon\}\}$.

A curve over $\mathbb{K}$ can be thought of as a family of curves depending on a parameter ϵ



$$\theta(\mathbf{z}) = \sum_{\mathbf{c}\in\mathbb{Z}^g} \exp\left[\frac{1}{2}\mathbf{c}^T B\mathbf{c} + \mathbf{c}^T\mathbf{z}\right] \quad \rightsquigarrow \quad \theta_{\mathscr{C}}(\mathbf{x}) = \sum_{\mathbf{c}\in\mathscr{C}} a_{\mathbf{c}} \exp[\mathbf{c}^T\mathbf{z}]$$

Main Idea

We study solutions to the KP equations arising from algebraic curves defined over a non-archimedean field $\mathbb{K}$, like $\mathbb{Q}(\epsilon)$ or $\mathbb{C}\{\{\epsilon\}\}$.

For $\epsilon \to 0$

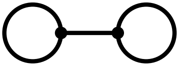
- The theta function becomes a finite sum of exponentials
- The function

$$p(x,y,t) = 2\frac{\delta^2}{\delta x^2}\log\tau(x,y,t)$$

  becomes a soliton solution of the KP equation

Degenerations of Theta Functions

Let X be a smooth curve of genus g over $\mathbb{K}$. The metric graph is $\mathrm{Trop}(X)$.



The metric graph $\Gamma = (V, E)$ of a genus 2 hyperelliptic curve

$$H_1(\Gamma, \mathbb{Z}) = \langle \gamma_1, \ldots, \gamma_g \rangle$$

is a free abelian group of rank g

- $e := |E|$
- Λ := $g \times e$ matrix whose i-th row records the coordinate of γ_i with respect to the standard basis of $\mathbb{Z}^e$
- Δ := diagonal $e \times e$ matrix that records edge lengths of the metric graph.

Definition

The Riemann matrix of $\Gamma = (V, E)$ is

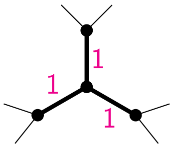
$$Q = \Lambda \Delta \Lambda^T$$

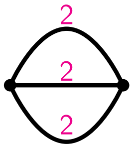
Example (g=2)

Consider $X := \{ y^2 = f(x) \}$ where

$$f(x) = (x-1)(x-1-\epsilon)(x-2)(x-2-\epsilon)(x-3)(x-3-\epsilon)$$

The six roots determine a subtree with six leaves which has a unique hyperelliptic covering by a metric graph of genus 2





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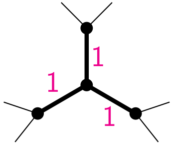

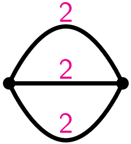

From the graph we can read off the tropical Riemann matrix Q

$$Q = \Lambda\Delta\Lambda^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$$

Degenerations of Theta Functions

Consider

$$B_\epsilon = -\frac{1}{\epsilon} Q + R(\epsilon)$$

Fix $\mathbf{a} \in \mathbb{R}^g$

$$\theta(\mathbf{z} + \frac{1}{\epsilon} Q\mathbf{a} \,|\, B_\epsilon) \; = \; \sum_{\mathbf{c}\in\mathbb{Z}^g} \exp\left[-\frac{1}{2\epsilon}\mathbf{c}^T Q\mathbf{c} + \frac{1}{\epsilon}\mathbf{c}^T Q\mathbf{a}\right] \cdot \exp\left[\frac{1}{2}\mathbf{c}^T R(\epsilon)\mathbf{c} + \mathbf{c}^T\mathbf{z}\right]$$

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Let $\epsilon \to 0$. This converges provided

$$\mathbf{c}^T Q\mathbf{c} - 2\mathbf{c}^T Q\mathbf{a} \geq 0 \text{ for all } \mathbf{c} \in \mathbb{Z}^g$$

or equivalently

$$\mathbf{a}^T Q\mathbf{a} \leq (\mathbf{a}-\mathbf{c})^T Q(\mathbf{a}-\mathbf{c}) \quad \text{for all } \mathbf{c} \in \mathbb{Z}^g$$

Voronoi and Delaunay

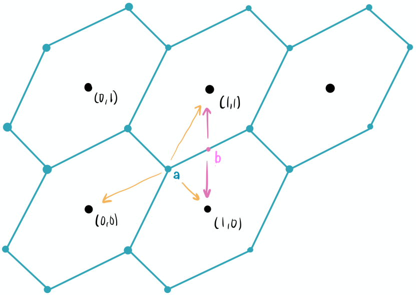
The condition

$$\mathbf{a}^T Q\mathbf{a} \leq (\mathbf{a}-\mathbf{c})^T Q(\mathbf{a}-\mathbf{c}) \quad \text{for all } \mathbf{c} \in \mathbb{Z}^g$$

holds if and only if $\mathbf{a}$ belongs to the *Voronoi cell* for Q

For $\mathbf{a}$ in the Voronoi cell for Q,
consider the associated Delaunay set:

$$\mathscr{D}_{\mathbf{a},Q} = \left\{\mathbf{c} \in \mathbb{Z}^g : \mathbf{a}^T Q\mathbf{a} = (\mathbf{a}-\mathbf{c})^T Q(\mathbf{a}-\mathbf{c})\right\}$$



$$\theta(\mathbf{z}+\frac{1}{\epsilon}Q\mathbf{a}\,|\,B_\epsilon) \;=\; \sum_{\mathbf{c}\in\mathbb{Z}^g} \exp\left[-\frac{1}{2\epsilon}\mathbf{c}^T Q\mathbf{c} + \frac{1}{\epsilon}\mathbf{c}^T Q\mathbf{a}\right]\cdot \exp\left[\frac{1}{2}\mathbf{c}^T R(\epsilon)\mathbf{c} + \mathbf{c}^T\mathbf{z}\right]$$

Theorem (Agostini-Fevola-M.-Sturmfels, 2021)

Fix $\mathbf{a}$ *in the Voronoi cell of the tropical Riemann matrix* Q*. For* $\epsilon \to 0$*, the series*

$$\theta(\mathbf{z}+\frac{1}{\epsilon}Q\mathbf{a}\,|\,B_\epsilon)$$

converges to the theta function supported on the Delaunay set $\mathscr{C} = \mathscr{D}_{\mathbf{a},Q}$*, namely*

$$\theta_{\mathscr{C}}(\mathbf{x}) \;=\; \sum_{\mathbf{c}\in\mathscr{C}} a_{\mathbf{c}} \exp[\mathbf{c}^T\mathbf{z}], \quad \text{where} \quad a_{\mathbf{c}} = \exp\left[\frac{1}{2}\mathbf{c}^T R(0)\mathbf{c}\right]$$

Example (g=2)

Theorem (Agostini-Fevola-M.-Sturmfels, 2021)

Fix $\mathbf{a}$ *in the Voronoi cell of* Q *and let* $\mathscr{C} = \mathscr{D}_{\mathbf{a},Q}$ *be the Delaunay set. As* $\epsilon \to 0$,

$$\theta(\mathbf{z} + \frac{1}{\epsilon} Q\mathbf{a} \,|\, B_\epsilon) \to \theta_{\mathscr{C}}(\mathbf{x}) \;=\; \sum_{\mathbf{c}\in\mathscr{C}} a_{\mathbf{c}} \exp[\mathbf{c}^T\mathbf{z}],$$

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where $a_{\mathbf{c}} = \exp\left[\frac{1}{2}\mathbf{c}^T R(0)\mathbf{c}\right]$

Example

For $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\mathscr{C} = \mathscr{D}_{\mathbf{a},Q} = \{(0,0),(1,0),(0,1),(1,1)\}$$

The associated theta function is

$$\theta_{\mathscr{C}} = a_{00} + a_{10}\exp[z_1] + a_{01}\exp[z_2] + a_{11}\exp[z_1 + z_2]$$

The Hirota Variety

Let $\mathscr{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\} \subset \mathbb{Z}^g$

$$\theta_{\mathscr{C}}(\mathbf{z}) = a_1 \exp[\mathbf{c}_1^T \mathbf{z}] + a_2 \exp[\mathbf{c}_2^T \mathbf{z}] + \cdots + a_m \exp[\mathbf{c}_m^T \mathbf{z}]$$

Consider

$$\tau(x, y, t) = \theta_{\mathscr{C}}(\mathbf{u}x + \mathbf{v}y + \mathbf{w}t) = \sum_{i=1}^{m} a_i \exp\Big[\big(\sum_{j=1}^{g} c_{ij} u_j\big)\, x + \big(\sum_{j=1}^{g} c_{ij} v_j\big)\, y + \big(\sum_{j=1}^{g} c_{ij} w_j\big)\, t\Big]$$

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$$\tau(x,y,t) = \theta_{\mathscr{C}}(\mathbf{u}x+\mathbf{v}y+\mathbf{w}t) = \sum_{i=1}^{m} a_i \exp\Big[\big(\sum_{j=1}^{g} c_{ij}u_j\big)\,x + \big(\sum_{j=1}^{g} c_{ij}v_j\big)\,y + \big(\sum_{j=1}^{g} c_{ij}w_j\big)\,t\Big]$$

Definition

The Hirota variety $\mathscr{H}_{\mathscr{C}}$ consists of all points $\big(\mathbf{a}, (\mathbf{u}, \mathbf{v}, \mathbf{w})\big)$ in $(\mathbb{K}^*)^m \times \mathbb{WP}^{3g-1}$ such that $\tau(x,y,t)$ satisfies Hirota's differential equation

Polynomials defining the Hirota Variety

Remark

Hirota's differential equation can be written via the Hirota differential operators as

$$P(D_x, D_y, D_t)\tau \bullet \tau = 0$$

where $P(x, y, t) = x^4 - 4xt + 3y^2$ gives the soliton dispersion relation

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For any two indices k, ℓ in $\{1, \dots, m\}$

$$P_{k\ell}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \ := \ P\big((\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{u},\ (\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{v},\ (\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{w} \big)$$

is a hypersurface in $\mathbb{WP}^{3g-1}$

Polynomials defining the Hirota Variety

The polynomials defining $\mathcal{H}_{\mathcal{C}}$ correspond to points in

$$\mathcal{C}^{[2]} = \left\{ \mathbf{c}_k + \mathbf{c}_\ell \,:\, 1 \le k < \ell \le m \right\} \subset \mathbb{Z}^g$$

Definition

A point $\mathbf{d}$ in $\mathcal{C}^{[2]}$ is uniquely attained if there exists precisely one index pair (k, ℓ) such that $\mathbf{c}_k + \mathbf{c}_\ell = \mathbf{d}$. In that case, (k, ℓ) is a unique pair.

Polynomials defining the Hirota Variety

Theorem (Agostini-Fevola-M.-Sturmfels)

The Hirota variety $\mathcal{H}_{\mathcal{C}}$ is defined by the quartics

$$P_{k\ell}(\mathbf{u},\mathbf{v},\mathbf{w}) \;:=\; P\big((\mathbf{c}_k-\mathbf{c}_\ell)\cdot\mathbf{u},\, (\mathbf{c}_k-\mathbf{c}_\ell)\cdot\mathbf{v},\, (\mathbf{c}_k-\mathbf{c}_\ell)\cdot\mathbf{w}\big)$$

for all unique pairs (k,ℓ) and by the polynomials

$$\sum_{\substack{1\le k<\ell\le m \\ \mathbf{c}_k+\mathbf{c}_\ell=\mathbf{d}}} P_{k\ell}(\mathbf{u},\mathbf{v},\mathbf{w})\, a_k a_\ell$$

for all non-uniquely attained points $\mathbf{d}\in\mathcal{C}^{[2]}$.
If all points in $\mathcal{C}^{[2]}$ are uniquely attained then $\mathcal{H}_{\mathcal{C}}$ is defined by the $\binom{m}{2}$ quartics $P_{k\ell}(\mathbf{u},\mathbf{v},\mathbf{w})$.

Example (The Square)

Let $g = 2$ and $\mathscr{C} = \{0,1\}^2$

$$\mathscr{C}^{[2]} = \{(0,1),(1,0),(1,1),(1,2),(2,1)\}$$

There are four unique pairs (k,ℓ)

$$P_{13} = P_{24} = u_1^4 - 4u_1w_1 + 3v_1^2$$

$$P_{12} = P_{34} = u_2^4 - 4u_2w_2 + 3v_2^2$$

The point $\mathbf{d} = (1,1)$ is not uniquely attained in $\mathscr{C}^{[2]}$

$$P(u_1+u_2, v_1+v_2, w_1+w_2)\, a_{00}a_{11} \;+\; P(u_1-u_2, v_1-v_2, w_1-w_2)\, a_{01}a_{10}$$

For any point in $\mathscr{H}_{\mathscr{C}} \subset (\mathbb{K}^*)^4 \times \mathbb{WP}^5$, we can write $\tau(x,y,t)$ as a $(2,4)$-soliton

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Example (The Cube)

Let $g = 3$ and consider the tropical degeneration of a smooth plane quartic C to a rational quartic

$$\begin{aligned}\theta_{\mathscr{C}} &= a_{000} + a_{100}\exp[z_1] + a_{010}\exp[z_2] + a_{001}\exp[z_3] + a_{110}\exp[z_1 + z_2] \\ &\quad + a_{101}\exp[z_1 + z_3] + a_{011}\exp[z_2 + z_3] + a_{111}\exp[z_1 + z_2 + z_3].\end{aligned}$$

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We compute the Hirota variety in $(\mathbb{K}^*)^8 \times \mathbb{WP}^8$.
The set $\mathscr{C}^{[2]}$ consists of 19 points.

- 12 pts uniquely attained, one for each edge of the cube → quartics $u_j^4 - 4u_j w_j + 3v_j^2$, one for each edge direction $\mathbf{c}_k - \mathbf{c}_\ell$
- 6 pts attained twice → contribute 6 equations, one for each facet
- $(1,1,1)$ four times

$$\begin{aligned}\rightarrow & P(u_1 + u_2 + u_3,\ v_1 + v_2 + v_3,\ w_1 + w_2 + w_3)\, a_{000} a_{111} \\ & + P(u_1 + u_2 - u_3,\ v_1 + v_2 - v_3,\ w_1 + w_2 - w_3)\, a_{001} a_{110} \\ & + P(u_1 - u_2 + u_3,\ v_1 - v_2 + v_3,\ w_1 - w_2 + w_3)\, a_{010} a_{101} \\ & + P(-u_1 + u_2 + u_3,\ -v_1 + v_2 + v_3,\ -w_1 + w_2 + w_3)\, a_{100} a_{011}\end{aligned}$$

The g-Cube

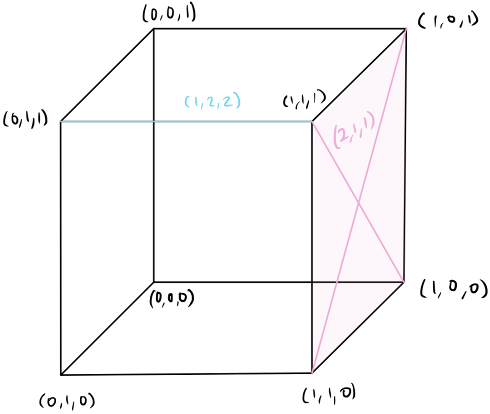
- Irreducible rational nodal curve with g nodes $\longrightarrow$ dual graph is a vertex with g loops $\longrightarrow$ Tropical Riemann matrix is I_g.
- fix the point $\mathbf{a} = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in \mathbb{R}^g \longrightarrow$ corresponding Delaunay set $\mathscr{C} = \mathscr{D}_{\mathbf{a},Q} = \{0,1\}^g$

$$\begin{aligned}\theta_{\mathscr{C}} = \quad & a_{00\ldots0} + a_{10\ldots0}\exp[z_1] + a_{010\ldots0}\exp[z_2] + \cdots + a_{0\ldots01}\exp[z_g] + \\ & + a_{110\ldots0}\exp[z_1+z_2] + a_{1010\ldots0}\exp[z_1+z_3] + a_{0\ldots011}\exp[z_{g-1}+z_g] + \\ & + \cdots + a_{11\ldots1}\exp[z_1+z_2+\cdots+z_g].\end{aligned}$$

For the equations cutting out the Hirota Variety, we are interested in the combinatorics of $\mathscr{C}^{[2]}$

The g-Cube: $\mathscr{C}^{[2]}$

One can observe that $\mathscr{C}^{[2]}$ is the set of lattice points in $2\mathrm{conv}(\mathscr{C})$ that are not vertices, so there are $3^g - 2^g$ points.
Each d–dimensional face of $\mathrm{conv}\mathscr{C}$ corresponds to a point that is attained 2^{d-1} times.



The Hirota Variety for the g-cube

We want to study the variety $\mathcal{H}_{\mathcal{C}}$ and the parametrization of the points $\big(\mathbf{a}, (\mathbf{u}, \mathbf{v}, \mathbf{w})\big) \in \mathcal{H}_{\mathcal{C}}$ in the parameter space $(\mathbb{K}^*)^{2^g} \times \mathbb{WP}^{3g-1}$ such that

$$\tau(x,y,t) = \theta_{\mathcal{C}}(\mathbf{u}x+\mathbf{v}y+\mathbf{w}t) = \sum_{i=1}^{2^g} a_i \exp\Big[\big(\sum_{j=1}^{g} c_{ij}u_j\big)\,x + \big(\sum_{j=1}^{g} c_{ij}v_j\big)\,y + \big(\sum_{j=1}^{g} c_{ij}w_j\big)\,t\Big].$$

can be expressed as a (k,n)-soliton.

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we want to write τ in the following form:

$$\sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{\substack{i,j \in I \\ i<j}} (\kappa_j - \kappa_i) \cdot \exp\Big[x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3\Big],$$

where the p_I are Plücker coordinates.

Expressing the g-cube τ-function as a $(g,2g)$-soliton

Lemma

The set $\mathscr{C}^{[2]}$ contains $g\cdot 2^{g-1}$ points which are uniquely attained. These contribute as generators of the Hirota variety $\mathscr{H}_{\mathscr{C}}$ with g quadrics of the form $u_j^4-4u_jw_j+3v_j^2$ for $j=1,\ldots,g$.

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Proof.

A point $\mathbf{c} \in \mathbb{C}$ is uniquely attained any time that the points $\mathbf{c}_k, \mathbf{c}_\ell \in \mathscr{C}$ such that $\mathbf{c}_k + \mathbf{c}_\ell = \mathbf{c}$ lie on same edge of the cube. It follows that such points will contribute the quartics

$$P_{k\ell}(\mathbf{u}, \mathbf{v}, \mathbf{w}) \; := \; P\big((\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{u}, (\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{v}, (\mathbf{c}_k - \mathbf{c}_\ell) \cdot \mathbf{w} \big).$$

The difference $\mathbf{c}_k - \mathbf{c}_\ell$ corresponds to the direction of the edge. Hence out of these $g2^{g-1}$ quartics of this form, g of them are distinct. □

The Parameterization

These g distinct equations induce a parametrization on the coordinates of $\mathbb{WP}^{3g-1}$ given by:

$$\begin{array}{c} u_1 = \kappa_1 - \kappa_2\,, \quad v_1 = \kappa_1^2 - \kappa_2^2, \quad w_1 = \kappa_1^3 - \kappa_2^3, \\ u_2 = \kappa_3 - \kappa_4\,, \quad v_2 = \kappa_3^2 - \kappa_4^2, \quad w_2 = \kappa_3^3 - \kappa_4^3, \\ \vdots \\ u_g = \kappa_{2g-1} - \kappa_{2g}\,, \quad v_g = \kappa_{2g-1}^2 - \kappa_{2g}^2, \quad w_g = \kappa_{2g-1}^3 - \kappa_{2g}^3. \end{array} \tag{1}$$

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Claim

The main component of the Hirota variety is a $3g$-dimensional irreducible component with a parametric representation

$$\begin{gathered} u_i = \kappa_{2i-1} - \kappa_{2i}, \quad v_i = \kappa_{2i-1}^2 - \kappa_{2i}^2, \quad w_i = \kappa_{2i-1}^3 - \kappa_{2i}^3, \quad \textit{for all } i = 1, \ldots, g \\ a_{\mathbf{c}} = \lambda_0 \prod_{i<j\in I} (\kappa_i - \kappa_j) \prod_{i:c_i=1} \lambda_i \ \textit{ where } I = \{2i : c_i = 0\} \cup \{2i-1 : c_i = 1\}, \textit{for all } \mathbf{c} \in \mathscr{C}. \end{gathered} \tag{2}$$

Soliton Matrix

With the parameterization in the previous slides, we can express

$$\tau(x,y,t) = \sum_{I \in \binom{[n]}{k}} p_I \cdot \prod_{\substack{i,j \in I \\ i<j}} (\kappa_j - \kappa_i) \cdot \exp\left[x \cdot \sum_{i \in I} \kappa_i + y \cdot \sum_{i \in I} \kappa_i^2 + t \cdot \sum_{i \in I} \kappa_i^3 \right],$$

with the $p_I = \lambda_0 \prod_{i: c_i = 1} \lambda_i$ for each I obtained from the points $\mathbf{c} \in \mathscr{C}$ by taking the set $I = \{2i : c_i = 0\} \cup \{2i - 1 : c_i = 1\}$

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The corresponding soliton matrix is the $g \times 2g$ matrix

$$\begin{pmatrix} \lambda_0\lambda_1 & \lambda_0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \lambda_g & 1 \end{pmatrix}$$

Works featured

- KP Solitons from Tropical Limits with Daniele Agostini, Claudia Fevola, and Bernd Sturmfels (arxiv 2101.10392)
  - Code can be found at https://mathrepo.mis.mpg.de/KPSolitonsFromTropicalLimits/index.html
- The Hirota Variety of the g-Cube - ongoing collaboration with Claudia Fevola.

Thank you!