Week 8: (uniform) convergence of functions.

Motivation: approximation of functions by Taylor series/Fourier series/polynomial etc. in computer. Can you approximate the derivative/integration/critical points/max value etc. at the same time? Can you preserve property of the approximation on a set $S$?

**Def.** A sequence of functions $(f_n)$ s.t. each $f_n$ is defined on $S$.

Remark: for any fixed $x_0 \in S$, $(f_n(x_0))$ is a sequence.

**Def.** Let $(f_n)$ be a sequence of functions on $S$.

$(f_n)$ converges pointwise to $f$ on $S$ if

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \forall x \in S.$$

Denote by $\lim_{n \to \infty} f_n = f$ (pointwise)

or $f_n \to f$ (pointwise).

**Example:**

- $f_n(x) = \frac{x}{n}, \quad S = \mathbb{R}, \quad f(x) = x$.

- $f_n(x) = x^n, \quad S = [0,1], \quad f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$
remark: \( f_n \to f \) pointwise translates to:

\[ \forall x \in S, \exists A > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, |f_n(x) - f(x)| < \varepsilon \]

Here the choice of \( N \) depends on \( \varepsilon \) and \( x \).

See example:

1. \( f_n(x) = x^n \) on \([0,1] \)

    for any \( x \in (0,1) \): \( \forall A > 0, \exists N = \log_x \frac{1}{A} \)

    \[ |f_n(x) - 0| = x^n < x^N \leq 3 \]

    In this example you cannot remove the dependence on \( x \).

2. \( f_n(x) = x - \frac{1}{n} \) on \( \mathbb{R} \)

    for any \( x \in \mathbb{R} \), \( \forall A > 0, \exists N = \frac{1}{A} \)

    \[ |f_n(x) - f(x)| = \frac{1}{n} \leq \frac{1}{N} = 3 \]

    Here \( N \) does not depend on \( x \).

Def: A sequence of functions \( (f_n) \) converges uniformly to \( f \) on \( S \)

if \( \forall A > 0, \exists N(\varepsilon) > 0 \text{ s.t. } \forall x \in S, \forall n \geq N, |f_n(x) - f(x)| < \varepsilon \).

denote \( f_n \to f \) uniformly \( \Rightarrow \lim_{n \to \infty} f_n = f \) uniformly.

Remark:

- \( f_n \to f \) uniformly \( \Rightarrow \) \( f_n \to f \) pointwise.

- \( f_n \) is contained within the \( \varepsilon \)-strip of \( f \) for all \( n \geq N \).
\[ f_n(x) = x - \frac{1}{n} \quad f_n \to f \text{ uniformly} \]

\[ f_n(x) = \frac{1}{n} \sin(nx) \quad f(x) = 0 \quad \text{on } \mathbb{R} \]

Show uniform convergence by definition

\[ \forall \varepsilon > 0 \quad \exists N = \frac{\varepsilon}{3} > 0 \quad \text{s.t. } \forall x \in \mathbb{R}, \text{ and } n > N \]

\[ |f_n(x) - f(x)| = \left| \frac{1}{n} \sin(nx) \right| \leq \frac{1}{n} < \frac{1}{N} = \frac{\varepsilon}{3} \]

\[ f_n(x) = x^n \quad f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & x = 1 \end{cases} \quad f_n \to f \text{ pointwise not uniformly} \]

Show by definition

\[ \exists \varepsilon = \frac{1}{2} \quad \forall N, \quad \exists x \in [0, 1] \text{ and } n > N \]

\[ s.t. \quad |f_n(x) - f(x)| = x^n = x^{n+1} \geq \frac{1}{2} > \varepsilon \]

This can be proved in an easier way.

Theorem: Suppose \( f_n \to f \) uniformly on \( S \). If \( f_n \) is continuous at \( x_0 \in S \) for all \( n \), then \( f \) is continuous at \( x_0 \in S \).

(Uniform convergence preserves continuity)

Application: Show now uniform convergence

\[ f_n(x) = x^n \text{ on } [0, 1] \quad f(x) = \begin{cases} 0 & x \in (0, 1) \\ 1 & x = 0 \end{cases} \text{ continuous on } [0, 1] \]

If \( f_n \to f \) uniformly then \( f \) is continuous on \([0, 1]\).

However, \( f \) is not continuous at \( x_0 = 1 \). So not uniformly convergent.

Show some function is continuous.

\[ \text{e.g. } f(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k^2} x^k \quad x \in [0, 1] \]
Proof: (Very classical \(\frac{S}{3}\) method!)

What we need: \(\left| f(x) - f(x_0) \right| \)

\[\forall \epsilon > 0 \text{ s.t. } \forall x \in S \text{ with } |x - x_0| < \delta, \text{ we have} \]
\[|f(x) - f(x_0)| < \epsilon \text{.} \quad \epsilon \text{ we do not have too much info,}
\]

but "bridge" through \(f_n\).

\[|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \]

want \(< \frac{\epsilon}{3} \quad + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \)

\[\text{Uniform Convergence} \quad \text{Continuity} \]

\[f_n \to f \text{ uniformly } \Rightarrow \exists N \text{ s.t. } \forall n > N, \forall x \in S, |f(x) - f(x)| < \frac{\epsilon}{3}. \]

In particular, \(n = N + 1\), \(|f_{N+1}(x) - f(x)| < \frac{\epsilon}{3}, \forall x \in S. \)

\[f_{N+1} \text{ is continuous } \Rightarrow \exists \delta > 0 \text{ s.t. } \forall x \in S \text{ with } |x - x_0| < \delta, \]
\[|f_{N+1}(x) - f_{N+1}(x_0)| < \frac{3}{5}. \]

then \(\forall x \in S \text{ with } |x - x_0| < \delta, \)
\[|f(x) - f(x_0)| < \epsilon. \]

\[\text{This shows: (under uniform convergence) continuity can pass through taking limit.} \]

Take any \((x_k) \subset S \), \(\lim_{k \to \infty} x_k = x_0 \), \(\lim_{k \to \infty} f(x_k) = f(x_0) \)

What we have is \(\lim_{k \to \infty} f(x_k) = \lim_{n \to \infty} \lim_{k \to \infty} f_n(x_k) = \lim_{n \to \infty} \lim_{k \to \infty} f_n(x_k) \).

(If we do not converge uniformly, then this is not true.)
Another example of exchange of limits: exchange with sup.

If $f_n \to f$ on $S$ uniformly, then $|f(x)|$.

$$\lim \sup_{n \to \infty} |f_n(x)| : x \in S = \sup \{ \lim_{n \to \infty} |f_n(x)| : x \in S \}$$

**Proof.** If $\sup_{x \in S} |f(x)| = +\infty$,

i.e. $\forall M > 0 \exists x \in S$ s.t. $|f(x)| > M$. $\sup_{x \in S} \lim_{n \to \infty} |f_n(x)| > M$. i.e. $|f_n(x)| > M - \epsilon$. $\sup_{x \in S} |f_n(x)| = M - \epsilon$. $\forall n > N$, $\forall x \in S$.

This holds for any $\epsilon > 0$. Similar strategy.

If $\sup_{x \in S} |f(x)| = a + \epsilon$. $\forall x \in S$, $\forall N_1 > 0$, $\exists N_1$ s.t. $\forall n > N_1$, $|f_n(x)| < a + \epsilon$. $\forall x \in S$.

On the other hand, $\forall \epsilon > 0$, $\exists x_0 \in S$ s.t. $|f(x_0)| \geq a - \frac{\epsilon}{2}$.

$\exists N_2$ s.t. $\forall n > N_2$, $|f_n(x_0) - f(x_0)| < \frac{\epsilon}{2}$ $\Rightarrow |f_n(x_0)| > a - \epsilon$.

$\Rightarrow \forall n > N_2$, $\sup |f_n| \geq a - \epsilon$.

Take $N = \max \{ N_1, N_2 \}$, $\forall n > N$ $\Rightarrow \sup |f_n| - a \leq \epsilon$. $\Rightarrow$

$\lim \sup_{n \to \infty} |f_n(x)| = a$.

Again, only pointwise convergence would not give this statement. (hw)

lim sup $= \sup \lim$ uniform convergence

example: $f_n = x^n$ on $[0, 1]$. 
A stronger statement:

$$f_n \to f \text{ uniformly on } S \iff \lim_{n \to \infty} \sup_{x \in S} |f_n(x) - f(x)| = 0.$$  

(if \(f(x) = 0\) in the limit, then the previous statement is 

$$\lim_{n \to \infty} \sup_{x \in S} |f_n| = \sup_{x \in S} |f| = 0. \text{ so } \Rightarrow\) .

Application: If \(f_n(x) \to 0\) pointwise.

then uniform convergence \(\iff\) \(\sup_{x \in S} |f_n(x)| \to 0\) as \(n \to \infty\).

Example: \(f_n(x) = \frac{x}{1 + nx^2}\) \(\sqrt{\text{,}}\) \(f_n(x) = n^2 x^n (1-x)\). \(X:\) (see textbook)

Exchange of limits with differentiation and integration:

If \(f_n \to f\) uniformly \((*)\) then 

$$\lim_{n \to \infty} \frac{d}{dx} f_n = \frac{d}{dx} \lim_{n \to \infty} f_n$$

and 

$$\lim_{n \to \infty} \int f_n = \int (\lim_{n \to \infty} f_n).$$

\((*)\): Needs extra condition that will be discussed later.