MATH 104 HOMEWORK 9 SOLUTIONS

(1) Ross 19.8:
(a) Use the Mean Value theorem to prove
\[ |\sin x - \sin y| \leq |x - y| \]
for all \( x, y \in \mathbb{R} \).

(b) Show \( \sin x \) is uniformly continuous on \( \mathbb{R} \).

Solutions:
(a) If \( x = y \), then we have \( 0 \leq 0 \). Otherwise assume \( x \neq y \) since \( x \) and \( y \) are symmetric in the expression. By Mean Value Theorem, for \( x, y \in \mathbb{R} \), there exists \( a \in (x, y) \) such that 
\[ f'(a) = \frac{f(y) - f(x)}{y - x}, \]
hence 
\[ \cos(a) = \frac{\sin(y) - \sin(x)}{y - x}. \]
Therefore
\[ |\sin y - \sin x| = |(y - x) \cos a| \leq |y - x|. \]

(b) This directly follows from part (a). For any \( \epsilon > 0 \), let \( \delta = \epsilon \), then for any \( x, y \in \mathbb{R} \) with \( |x - y| < \delta \), we have
\[ |f(x) - f(y)| \leq |x - y| < \delta = \epsilon. \]
So this proves uniform continuity on \( \mathbb{R} \).

(2) Ross 28.3
(b) Let \( f(x) = x^{1/3} \) for \( x \in \mathbb{R} \) and use the definition of derivative to prove \( f'(x) = \frac{1}{3}x^{-2/3} \) for \( x > 0 \).
(c) Is the function \( f \) in part (b) differentiable at \( x = 0 \)? Explain.

Solutions:
(b) When \( y, x \neq 0 \) and \( x \neq y \), we have
\[ \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{y \to x} \frac{\frac{1}{y^{2/3}}}{y^{2/3} + y^{1/3}x^{1/3} + x^{2/3}} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}. \]
\( f \) is not differentiable at \( x = 0 \). Consider the following limit as \( y \to 0 \):
\[ \frac{f(y) - f(0)}{y - 0} = \frac{y^{1/3}}{y} = \frac{1}{y^{2/3}}. \]
Since \( \lim_{y \to 0} y^{-2/3} \) does not exist (one can use a sequence \( y_n = \frac{1}{n} \) and see \( y_n^{-2/3} \) goes to \( \infty \)), so \( f \) is not differentiable at \( x = 0 \).

(3) (20 pt) Ross 28.4: Let \( f(x) = x^2 \sin \frac{1}{x} \) for \( x \neq 0 \) and \( f(0) = 0 \).
(a) Use Theorems 28.3 and 28.4 to show \( f \) is differentiable at each \( a \neq 0 \) and calculate \( f'(a) \). Use, without proof, the fact that \( \sin x \) is differentiable and that \( \cos x \) is its derivative.
(b) Use the definition to show \( f \) is differentiable at \( x = 0 \) and \( f'(0) = 0 \).
(c) Show \( f' \) is not continuous at \( x = 0 \).

**Solutions:**

(a) Since \( \sin x \) is differentiable at any \( x \in \mathbb{R} \) and \( \frac{1}{x} \) is differentiable at \( x \neq 0 \), by Chain rule we have that \( \sin \frac{1}{x} \) is differentiable at \( a \neq 0 \), and

\[
\left( \sin \frac{1}{x} \right)'(a) = -\cos \left( \frac{1}{a} \right) \frac{1}{a^2}.
\]

And since \( x^2 \) is differentiable at any \( x \neq 0 \), then using product of differentiable functions \( x^2 \sin \frac{1}{x} \) is differentiable at \( x \neq 0 \). And by product rule

\[
(x^2 \sin \frac{1}{x})'(a) = 2a \sin \frac{1}{a} + a^2 \cdot (-\cos \frac{1}{a} \frac{1}{a^2}) = 2a \sin \frac{1}{a} - \cos \frac{1}{a}.
\]

(b) Since \( f'(y) - f(0) = \frac{y^2 \sin \frac{1}{y}}{y-0} = y \sin \frac{1}{y} \), therefore

\[
\lim_{y \to 0} \frac{f(y) - f(0)}{y - 0} = \lim_{y \to 0} y \sin \frac{1}{y} = 0.
\]

For the last equality see Exercise 17.9(c) in previous homework. So \( f \) is differentiable at \( x = 0 \) and \( f'(0) = 0 \).
(c) Consider the function given by \( f'(a) = 2a \sin \frac{1}{a} - \cos \frac{1}{a} \) for \( a \neq 0 \) and \( f'(0) = 0 \), we show that \( f' \) is not continuous at \( a = 0 \). Take a sequence \( a_n = \frac{1}{\pi n} \), then \( a_n \to 0 \) and \( f'(a_n) = -1 \) for all \( n \). So \( \lim_{n \to \infty} f'(a_n) = -1 \neq f'(0) \). Therefore \( f' \) is not continuous at 0.

(4) (20 pt) Ross 28.8: Let \( f(x) = x^2 \) for \( x \) rational and \( f(x) = 0 \) for \( x \) irrational.
(a) Prove \( f \) is continuous at \( x = 0 \).
(b) Prove \( f \) is discontinuous at all \( x \neq 0 \).
(c) Prove \( f \) is differentiable at \( x = 0 \).

**Solutions:**

(a) For any \( \epsilon > 0 \), let \( \delta = \sqrt{\epsilon} > 0 \), then for any \( x \in \mathbb{R} \) with \( |x - 0| < \delta \), we have either \( x \in \mathbb{Q} \) or \( x \notin \mathbb{Q} \). If \( x \in \mathbb{Q} \), then

\[
|f(x) - f(0)| = |x^2| < \delta^2 = \epsilon.
\]

If \( x \notin \mathbb{Q} \), then

\[
|f(x) - f(0)| = 0 < \epsilon.
\]

Therefore \( f \) is continuous at \( x = 0 \).
(b) For any \( x_0 \neq 0 \), take two sequences \( (a_n) \) and \( (b_n) \), such that \( a_n \in \mathbb{Q} \) and \( b_n \notin \mathbb{Q} \) for all \( n \) and \( \lim a_n = \lim b_n = x_0 \). Then

\[
\lim f(a_n) = \lim a_n^2 = x_0^2 \neq 0
\]
but
\[
\lim_{x \to b_n} f(x) = \lim_{x \to 0} = 0.
\]

So \( f \) is not continuous at \( x_0 \).

(c) Consider \( f(x) = x^2 - \frac{1}{1-x^2} \) for \( x \neq 0 \). If \( x \in \mathbb{Q} \), then \( \lim_{x \to 0} \frac{f(x) - f(0)}{x-0} = \frac{x^2}{x} = x \); if \( x \notin \mathbb{Q} \), then
\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x-0} = \frac{0}{0} = 0 \quad (\text{note here } x \neq 0!).
\]
Therefore
\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x-0} = 0.
\]

So \( f \) is differentiable at \( x = 0 \).

(5) (20 pt) Ross 29.1: Determine whether the conclusion of the Mean Value Theorem holds for the following functions on the specified intervals. If the conclusion holds, give an example of a point \( x \) satisfying (1) of Theorem 29.3. If the conclusion fails, state which hypotheses of the Mean Value Theorem fail.

(a) \( x^2 \) on \([-1, 2]\);
(b) \( \sin x \) on \([0, \pi]\);
(c) \( |x| \) on \([-1, 2]\);
(d) \( \frac{1}{x} \) on \([-1, 1]\).

**Solutions:**

(a) \( f'(1/2) = 1 = \frac{2^2 - (-1)^2}{2 - (-1)} \); 
(b) \( f'(\pi/2) = 0 = \frac{\sin(0) - \sin(\pi)}{0 - \pi} \);
(c) \( f \) is not differentiable at \( x = 0 \in (-1, 2) \), so the hypothesis “\( f \) is differentiable on \( (a, b) \)” fails.
(d) \( f \) is not continuous at \( x = 0 \in [-1, 1] \), so the hypothesis “\( f \) is continuous on \( [a, b] \)” fails.

(6) Ross 29.4: Let \( f \) and \( g \) be differentiable functions on an open interval \( I \). Suppose \( a, b \) in \( I \) satisfy \( a < b \) and \( f(a) = f(b) = 0 \). Show \( f'(x) + f(x)g'(x) = 0 \) for some \( x \in (a, b) \). Hint: Consider \( h(x) = f(x)e^{g(x)} \).

**Solutions:** Consider \( h(x) = f(x)e^{g(x)} \), since both \( f \) and \( g \) are differentiable, by product rule and chain rule, \( h(x) \) is differentiable on \( I \), and
\[
h'(x) = f'(x)e^{g(x)} + f(x)e^{g(x)}g'(x) = e^{g(x)}(f'(x) + f(x)g'(x)).
\]
Since \( h(a) = h(b) = 0 \), by mean value theorem, there exists \( x \in (a, b) \) such that
\[
h'(x) = \frac{h(a) - h(b)}{a - b} = 0
\]
and therefore this \( x \) satisfies \( f'(x) + f(x)g'(x) = 0 \).

(7) Ross 29.5: Let \( f \) be defined on \( \mathbb{R} \), and suppose \(|f(x) - f(y)| \leq (x - y)^2\) for all \( x, y \in \mathbb{R} \). Prove \( f \) is a constant function.

**Solutions:** We show that \( f \) is differentiable and \( f'(x) = 0 \) for any \( x \in \mathbb{R} \). Fix \( x \in \mathbb{R} \) and consider \( \frac{f(x) - f(y)}{x - y} \) for any \( y \neq x \). Since \(|f(x) - f(y)| \leq |x - y|^2\) we have
\[
\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x - y|.
\]
And since \( \lim_{y \to x} |x - y| = 0 \) by comparison we have that

\[
\lim_{y \to x} \left| \frac{f(x) - f(y)}{x - y} \right| = 0.
\]

And this shows \( f'(x) = 0 \). Since \( f'(x) = 0 \) everywhere, \( f \) is a constant function.