(1) Ross 14.1: Determine which of the following series converge. Justify your answers.

(d) \( \sum \frac{n!}{n^4+3} \)

Solutions:
(d) We use ratio test here: since
\[
\lim \left| \frac{a_{n+1}/a_n}{1} \right| = \lim \left| \frac{(n+1)!}{(n+1)^4+3} \cdot \frac{n!}{n^4+3} \right| = \lim (n+1) \frac{(n+1)^4+3}{n^4+3} = \lim (n+1) = +\infty
\]
Therefore \( \sum a_n \) diverges.

(e) \( \sum \frac{\cos^2 n}{n^2} \)
Solutions:
(e) Since \( \cos^2 n \leq 1 \), we apply the comparison test with Example 2:
\[
\sum |a_n| \leq \sum \frac{1}{n^2} < +\infty.
\]
Therefore \( \sum a_n \) converges.

(2) Ross 14.3: Determine which of the following series converge. Justify your answers.

(a) \( \sum \frac{1}{\sqrt{n}} \)

Solutions:
(a) We apply ratio test:
\[
\lim |a_{n+1}/a_n| = \lim \left| \frac{1}{\sqrt{(n+1)!}} / \frac{1}{\sqrt{n!}} \right| = \lim \frac{1}{\sqrt{n+1}} = 0
\]
Therefore \( \sum a_n \) converges.

(e) \( \sum \sin \left( \frac{n\pi}{9} \right) \)
Solutions:
(e) We first show that \( \lim a_n = 0 \) does not hold. Since \( a_{18k+3} = \sin \left( \frac{(18k+3)\pi}{9} \right) = \sin \frac{\pi}{2} \) is a nonzero constant, we find a subsequence of \( a_n \) that has a nonzero limit. Therefore \( \lim a_n = 0 \) does not hold, and by Corollary 14.5, \( \sum a_n \) diverges.

(3) Ross 14.4: Determine which of the following series converge. Justify your answers.

(b) \( \sum [\sqrt{n+1} - \sqrt{n}] \)
Solutions: We first notice the following equality:

\[ a_n = |\sqrt{n + 1} - \sqrt{n}| = \frac{\sqrt{n + 1} + \sqrt{n}}{\sqrt{n + 1} + \sqrt{n}} |\sqrt{n + 1} - \sqrt{n}| = \frac{1}{\sqrt{n + 1} + \sqrt{n}} \]

Then using comparison test and Example 2:

\[ \sum a_n \geq \sum \frac{1}{2\sqrt{n + 1}} = +\infty \]

we get that \( \sum a_n \) diverges.

(4) Ross 14.7: Prove that if \( \sum a_n \) is a convergent series of nonnegative numbers and \( p > 1 \), then \( \sum a_n^p \) converges.

Solutions: Since \( \sum a_n \) converges, by Corollary 14.5, \( \lim a_n = 0 \). Hence for \( \epsilon = \frac{1}{2} \), there exists \( N \) such that any \( n > N \) we have \( |a_n| < \epsilon \). Since \( a_n \geq 0 \), this is \( 0 \leq a_n < 1/2 \) for all \( n > N \). Since \( p > 1 \), \( 0 \leq a_n^{p-1} < 1 \), and hence multiplying by \( a_n \geq 0 \) to the inequality, \( 0 \leq a_n^p < a_n \) for all \( n > N \). Since \( \sum_{n=N+1}^{\infty} a_n \) converges, by comparison test, \( \sum_{n=N+1}^{\infty} a_n^p \) converges. And therefore adding finitely many terms, \( \sum_{n=1}^{\infty} a_n^p \) converges.

(5) Ross 14.12: Let \( (a_n)_{n \in \mathbb{N}} \) be a sequence such that \( \lim \inf |a_n| = 0 \). Prove there is a subsequence \( (a_{n_k})_{k \in \mathbb{N}} \) such that \( \sum_{k=1}^{\infty} |a_{n_k}| \leq 0 \).

Solutions: Since \( \lim \inf |a_n| = 0 \), by Theorem 11.7, there is a subsequence \( (a_{n_k}) \) such that \( \lim_{N \to \infty} |a_{n_k}| = 0 \).

We extract a subsequence of \( |a_{n_N}| \) by induction. For \( \epsilon_1 = 1 \), there exists \( N_1 \) such that any \( N \geq N_1 \) we have \( |a_{n_N}| < \epsilon_1 = 1 \). In particular, \( |a_{n_N_1}| < 1 \). Apply the same argument, there exists \( N_2 \) such that for any \( N \geq N_2 \), \( |a_{n_N}| < \frac{1}{2} \). If \( N_2 \leq N_1 \), replace \( N_2 \) by \( N_1 + 1 \). Therefore we have \( |a_{n_{N_k}}| < 1/2 \), \( N_2 > N_1 \). Now assume \( a_{n_{N_k}} \) is fixed for \( i = 1, \ldots, k \), then for \( k + 1 \), we find \( a_{n_{N_{k+1}}} \) by the following. For \( \epsilon_{k+1} = 2^{-(k+1)} \), there exists \( N_{k+1} \) such that for all \( N \geq N_{k+1} \), \( |a_{n_N}| < \epsilon_{k+1} \). If \( N_{k+1} \leq N_k \), replace \( N_{k+1} \) by \( N_k + 1 \). So we have \( |a_{n_{N_{k+1}}}| < 2^{-(k+1)} \), \( N_{k+1} > N_k \).

By induction, we have a subsequence \( \{a_{n_{N_k}}\}_{k \in \mathbb{N}}, |a_{n_{N_k}}| < 2^{-k} \). Since a subsequence of a subsequence is still a subsequence, we define the subsequence \( (a_{n_k}) \) by taking \( n_k = n_{N_k} \) for all \( k \).

Since the geometric series \( \sum_{k \in \mathbb{N}} 2^{-k} \) converges, by comparison test, \( \sum_{k} a_{n_k} \) also converges (in fact it converges absolutely).

(6) Ross 14.13:

(c) Prove \( \sum_{n=1}^{\infty} \frac{n-1}{2^n} = \frac{1}{2} \). Hint: Note \( \frac{k-1}{2^k+1} = \frac{k}{2^k} - \frac{k+1}{2^k+1} \).

(d) Using (c) to calculate \( \sum_{n=1}^{\infty} \frac{n}{2^n} \).

Solutions:
(c) Using the hint, we rewrite the sum and using cancellation
\[
\sum_{n=1}^{\infty} \frac{n - 1}{2^n+1} = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{n - 1}{2^n+1} = \lim_{N \to \infty} \sum_{n=1}^{N} \left( \frac{n}{2^n} - \frac{n+1}{2^{n+1}} \right) = \lim_{N \to \infty} \left( \frac{1}{2} - \frac{N+1}{2^{N+1}} \right) = \frac{1}{2}.
\]
Here we used \( \lim_{N \to \infty} \frac{N+1}{2^{N+1}} = 0 \).

(d) First of all we multiply (c) by 2 and get
\[
\sum_{n=1}^{\infty} \frac{n - 1}{2^n} = 1.
\]
Then we compute \( \sum_{n=1}^{\infty} \frac{n}{2^n} \) which can be written as the sum of two CONVERGENT series, where the first term is obtained above and the second term is a geometric series:
\[
\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} \frac{n - 1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + 1 = 2.
\]

(7) Ross 15.1: Determine which of the following series converge. Justify your answers.

(a) \( \sum \frac{(-1)^n}{n} \).

**Solutions:** This is a direct application of the alternating series test. Since \( \left( \frac{1}{n} \right) \) is a decreasing sequence that goes to 0, we know that \( \sum \frac{(-1)^n}{n} \) converges.

(8) Ross 15.4: Determine which of the following series converge. Justify your answers.

(a) \( \sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \log n} \)

(c) \( \sum_{n=1}^{\infty} \frac{1}{n(\log n)(\log \log n)} \)

**Solutions:**

(a) We apply the comparison test. Note that \( \lim_{n \to \infty} \frac{\log n}{\sqrt{n}} = 0 \), therefore there exists \( N \) such that \( \left| \frac{\log n}{\sqrt{n}} \right| < 1 \) for all \( n > N \). Then \( \frac{1}{\sqrt{n} \log n} > \frac{1}{n} \) for all \( n > N \). Since \( \sum \frac{1}{n} \) diverges, \( \sum \frac{1}{\sqrt{n} \log n} \) diverges.

(c) We use integral test. Consider the function \( f(x) = \frac{1}{x(\log x)(\log \log x)} \) on \( x \geq 4 \), which is a nonnegative decreasing function. Compute its integral
\[
\int_{4}^{\infty} f(x) \, dx = \int_{4}^{\infty} \frac{1}{x(\log x)(\log \log x)} \, dx = \log \log x \bigg|_{4}^{\infty} = +\infty.
\]
which diverges. Therefore the series diverges.