MATH 104 HOMEWORK 10 SOLUTIONS

(1) Ross 31.1: Find the Taylor series for \( \cos x \) and indicate why it converges to \( \cos x \) for all \( x \in \mathbb{R} \).

**Solutions:** We compute the derivative \( \cos^{(n)}(0) \). First we notice that

\[
\cos^{(n)}(x) = \begin{cases} 
\cos x & n = 4k \\
-\sin x & n = 4k + 1 \\
-\cos x & n = 4k + 2 \\
\sin x & n = 4k + 3 
\end{cases}
\]

Hence

\[
\cos^{(n)}(0) = \begin{cases} 
1 & n = 4k \\
0 & n = 2k + 1 \\
-1 & n = 2k + 2 
\end{cases}
\]

Therefore the Taylor series for \( \cos x \) at 0 is given by

\[
\frac{1}{0!} + \frac{-1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{-1}{6!}x^6 + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}.
\]

By changing variable \( x^2 = y \) we get a series

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}y^k
\]

Since all coefficients \( f^{(n)}(0) \) is bounded by 1, we have that the series converges everywhere to \( \cos x \).

(2) (20 pt) Ross 31.5: Let \( g(x) = e^{-1/x^2} \) for \( x \neq 0 \) and \( g(0) = 0 \).

(a) Show \( g^{(n)}(0) = 0 \) for all \( n = 0, 1, 2, 3, \ldots \).

(b) Show the Taylor series for \( g \) about 0 agrees with \( g \) only at \( x = 0 \).

**Solutions:**

(a) Using example 3 of section 31, we have shown that the derivative for \( g(y) = e^{-1/y} \) at 0 is given by

\[
g^{(n)}(0) = 0, \forall n
\]

And since \( f(x) = g(x^2) \), by chain rule, we have

\[
f'(x) = g'(x) \cdot 2x,
\]

\[
f^{(2)}(x) = g^{(2)}(x) \cdot 2x + 2g'(x),
\]

\[
f^{(3)}(x) = g^{(3)}(x) \cdot 2x + 4g^{(2)}(x), \ldots
\]
And we prove by induction that
\[ f^{(n)}(x) = g^{(n)}(x) \cdot 2x + 2(n-1)g^{(n-1)}(x). \]

First of all, this holds for \( n = 1 \). Suppose this is true for \( n \), then
\[
\begin{align*}
f^{(n+1)}(x) &= \frac{d}{dx} \left( f^{(n)}(x) \cdot 2x + 2(n-1)g^{(n-1)}(x) \right) \\
&= g^{(n)}(x) \cdot 2x + 2g^{(n)}(x) + 2(n-1)g^{(n-1)}(x) \\
&= g^{(n+1)}(x) \cdot 2x + 2ng^{(n)}(x)
\end{align*}
\]
which proves the induction. Therefore we have \( f^{(n)}(0) = g^{(n)}(0) \cdot 0 + 2(n-1)g^{(n-1)}(0) = 0 \) for all \( n \).

(b) The Taylor series for \( g \) is given by 0 from the previous question. So this agrees with \( g(0) = 0 \). And we next show that \( g(x) \neq 0 \) for any \( x \neq 0 \), which is obvious from the fact that \( e^t \neq 0, \forall t \in \mathbb{R} \).

(3) (20 pt) Ross 31.6: Assume \( x > 0 \), let \( M \) be as in the proof of Theorem 31.3, and let
\[
F(t) = f(t) + \sum_{k=1}^{n-1} \frac{(x-t)^k}{k!} f^{(k)}(t) + M \cdot \frac{(x-t)^n}{n!}
\]
for \( t \in [0, x] \).

(a) Show \( F \) is differentiable on \( [0, x] \) and
\[
F'(t) = \frac{(x-t)^{n-1}}{(n-1)!} [f^{(n)}(t) - M].
\]

(b) Show \( F(0) = F(x) \).

(c) Apply Rolle’s Theorem to \( F \) to obtain \( y \in (0, x) \) such that \( f^{(n)}(y) = M \).

**Solutions:**

(a) By product rule, \( F \) is differentiable on \( [0, x] \), since \( f(t), f^{(k)}(t) \) and \( (x-t)^k \) are all differentiable. And we compute the derivative
\[
F'(t) = f'(t) + \sum_{k=1}^{n-1} \left( \frac{-(x-t)^{k-1}}{(k-1)!} f^{(k)}(t) + \frac{(x-t)^k}{k!} f^{(k+1)}(t) \right) + M \cdot \frac{-(x-t)^{n-1}}{(n-1)!}
\]
\[
= f'(t) - f'(t) + \sum_{k=2}^{n-1} \frac{-(x-t)^{k-1}}{(k-1)!} f^{(k)}(t) + \sum_{k=1}^{n-1} \frac{-(x-t)^k}{k!} f^{(k+1)}(t) + M \cdot \frac{-(x-t)^{n-1}}{(n-1)!}
\]
\[
= \sum_{k=1}^{n-2} \frac{-(x-t)^{k}}{(k)!} f^{(k+1)}(t) + \sum_{k=1}^{n-1} \frac{-(x-t)^k}{k!} f^{(k+1)}(t) + M \cdot \frac{-(x-t)^{n-1}}{(n-1)!}
\]
\[
= \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t) + M \cdot \frac{-(x-t)^{n-1}}{(n-1)!}
\]
\[
= \frac{(x-t)^{n-1}}{(n-1)!} [f^{(n)}(t) - M].
\]
(b) We prove this by direct computation.

\[ F(0) = f(0) + \sum_{k=1}^{n-1} \frac{(x)^k}{k!} f^{(k)}(0) + M \cdot \frac{(x)^n}{n!} = F(x) \]

since \( M \) is given by this equality.

(c) Apply Rolle’s theorem, there exists \( y \in (0, x) \) such that \( F'(y) = 0 \). From part (a), this is the same as

\[ 0 = F'(y) = \frac{(x - y)^{n-1}}{(n-1)!} [f^{(n)}(y) - M]. \]

Since \( y < x \), we have \( f^{(n)}(y) = M \).

(4) Ross 32.2: Let \( f(x) = x \) for rational \( x \) and \( f(x) = 0 \) for irrational \( x \).

(a) Calculate the upper and lower Darboux integrals for \( f \) on the interval \([0, b]\).

(b) Is \( f \) integrable on \([0, b]\)?

**Solutions:**

(a) Consider any partition \( P = \{0 = t_0 < t_1 < \ldots < t_n = b\} \). Since irrational numbers are dense in any interval \([t_{k-1}, t_k]\), we have

\[ m(f, [t_{k-1}, t_k]) = 0, \forall k. \]

And since rational numbers are dense as well, we have (even when \( t_k \) is irrational!)

\[ M(f, [t_{k-1}, t_k]) = t_k. \]

since for any \( \epsilon > 0 \) there exists \( x \in \mathbb{Q} \) such that \( 0 \leq t_k - x < \epsilon \).

Therefore the upper Darboux sum is given by

\[ U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} t_k \cdot (t_k - t_{k-1}) \]

Take \( t_k = \frac{kb}{n} \), then

\[ U(f, P) = \sum_{k=1}^{n} \frac{kb}{n} \cdot \frac{b}{n} = \frac{b^2}{n^2} \frac{n(n+1)}{2} = \frac{b^2}{2} \frac{n+1}{n}. \]

Take \( n \to \infty \) we get \( U(f) \leq \frac{b^2}{2} \). And since for any partition \( P \), we have \( t_k > \frac{t_k + t_{k-1}}{2} \), hence

\[ U(f, P) = \sum_{k=1}^{n} t_k \cdot (t_k - t_{k-1}) > \sum_{k=1}^{n} \frac{t_k + t_{k-1}}{2} \cdot (t_k - t_{k-1}) = \frac{1}{2} (t_n^2 - t_0^2) = \frac{b^2}{2}. \]

So \( U(f) \geq \frac{b^2}{2} \). Combining the two inequalities we have

\[ U(f) = \frac{b^2}{2}. \]
On the other hand
\[
L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) = 0
\]
so
\[
L(f) = 0.
\]
(b) Since \( L(f) \neq U(f) \), \( f \) is NOT integrable on \([0, b]\).

(5) Ross 32.7: Let \( f \) be integrable on \([a, b]\), and suppose \( g \) is a function on \([a, b]\) such that \( g(x) = f(x) \) except for finitely many \( x \) in \([a, b]\). Show \( g \) is integrable and \( \int_a^b f = \int_a^b g \). Hint: First reduce to the case where \( f \) is the function identically equal to 0.

**Solutions:**

Let \( h = g - f \), then \( h = 0 \) except at finitely many points \( \{x_1, \ldots, x_j\} \).

If we can prove that \( h \) is integrable and \( \int_a^b h = 0 \), then \( g = h + f \) is the sum of two integrable functions, hence by Theorem 33.3 \( g \) is integrable and \( \int_a^b g = \int_a^b h + \int_a^b f = \int_a^b f \).

Take any partition \( P = \{a = u_0 < u_1 < \ldots < u_n = b\} \). Let \([u_{i_1}, u_{i_1+1}], \ldots, [u_{i_j}, u_{i_j+1}]\) be the intervals that contain \( x_1, \ldots, x_j \). Therefore

\[
L(h, P) = \sum_{k=1}^{n} m(h, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) \geq \sum_{k=1}^{j} -|f(x_k)|(t_{i_k+1} - t_{i_k})
\]

and

\[
U(h, P) = \sum_{k=1}^{n} m(h, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) \leq \sum_{k=1}^{j} |f(x_k)|(t_{i_k+1} - t_{i_k}).
\]

For any \( \epsilon \), there exists a partition \( P \) such that \( \sum_{k=1}^{j} (t_{i_k+1} - t_{i_k}) < \frac{\epsilon}{\max\{|f(x_1)|, \ldots, |f(x_j)|\}} \).

And for this partition
\[
-\epsilon < L(h, P) \leq U(h, P) < \epsilon.
\]

Since \( \epsilon \) can be arbitrarily small, we have
\[
0 = L(h) = U(h)
\]
and this shows \( h \) is integrable and \( \int_a^b h = 0 \).

(6) (20 pt) Ross 33.3: A function \( f \) on \([a, b]\) is called a step function if there exists a partition \( P = \{a = u_0 < u_1 < \cdots < u_m = b\} \) of \([a, b]\) such that \( f \) is constant on each interval \((u_{j-1}, u_j)\), say \( f(x) = c_j \) for \( x \in (u_{j-1}, u_j) \).

(a) Show that a step function \( f \) is integrable and evaluate \( \int_a^b f \).

(b) Evaluate the integral \( \int_0^4 P(x)dx \) for the postage-stamp function \( P \) in Exercise 17.16

**Solutions:**
(a) For any $\epsilon > 0$, take any partition $P$ such that $\text{mesh}(P) < \frac{\epsilon}{\sum_{j=1}^{m-1} |c_{j+1} - c_j|}$. Let $[v_j, v'_j]$ be the interval in $P$ that contains $u_j$, $j = 1, \ldots, m - 1$. Then

$$U(f, P) - L(f, P) \leq \sum_{j=1}^{m-1} |c_{j+1} - c_j|(v'_j - v_j) \leq \sum_{j=1}^{m-1} |c_{j+1} - c_j| \cdot \text{mesh}(P) < \epsilon$$

By Theorem 32.7, this implies that $f$ is integrable. And

$$\int_a^b f = \sum_{j=1}^{m} c_j (u_j - u_{j-1}).$$

(b) Directly compute the integral

$$\int_0^1 P(x)dx = A(1 - 0) + (A + B)(2 - 1) + (A + 2B)(3 - 2) + (A + 3B)(4 - 3) = 4A + 6B.$$

(7) Ross 33.4: Give an example of a function $f$ on $[0, 1]$ that is not integrable for which $|f|$ is integrable.

**Solutions:** Let $f$ be $-1$ at rational points and 1 at irrational points. Then for the same reason as Example 2 in section 32, $U(f) = 1$ and $L(f) = -1$, hence $f$ is not integrable. On the other hand, $|f| = 1$ for all $x$, so is a constant function. And therefore $|f|$ is integrable.