MATH 104 SECTION 6 MIDTERM SOLUTIONS

Read this first:
The midterm is closed book - no textbook, no notes, no calculators, etc. You will not be expected to know the proofs of the theorems by heart, but you should be comfortable with the definitions, know the theorem statements, and be able to apply the theorems to problems (such as you did in the homework).

The midterm will consist of a few multiple choice questions, followed by three longer questions that will require you to construct a proof. Your proof should be similar in rigour to those given in the textbook or in homework solutions.

There are four problems to this exam. Each problem has its point total and breakdown listed. The entire exam is out of 100 points.

For Problem 2-4, write your answer in the space provided in clear steps and provide necessary justifications. Please be sure to indicate if the solution to a particular problem appears on a different page so the grader can find it.

This exam is printed single-sided.

Do not tear any page!

Good luck!

Your Name:
CalNet ID:
Problem 1 (20 pt). Determine whether the following statements are true or false. Make sure to fill the whole circle of your choice. You do not need to provide any justification, and no explanation will be used for grading.

(1) Let $f(x)$ be given by $f(x) = 1 - x^2$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \notin \mathbb{Q}$. Then $f$ is continuous at $x = 1$.

True $\bigcirc$ False $\bigcirc$

Solution: True. One can show this by using $\varepsilon - \delta$ definition. For any $\varepsilon > 0$, let $\delta = \sqrt{\varepsilon + 1} - 1 > 0$ (so that $\delta^2 + 2\delta = \varepsilon$), then for any $x$ with $|x - 1| < \delta$, either $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}$. If such $x \in \mathbb{Q}$, then

$$|f(x) - f(1)| = |1 - x^2 - 0| = |-(x - 1)^2 - 2(x - 1)| \leq \delta^2 + 2\delta = \varepsilon.$$ 

If such $x \notin \mathbb{Q}$, then $|f(x) - f(1)| = |0 - 0| = 0 < \varepsilon$.

(2) The series $\sum_{n=1}^{\infty} \frac{n}{n^2 + (-1)^n}$ diverges.

True $\bigcirc$ False $\bigcirc$

Solution: True. One can use comparison with series $\sum \frac{1}{n}$. Since $\lim_{n \to \infty} \frac{n}{n^2 + (-1)^n} / \frac{1}{n} = 1$, and both series are nonnegative, the divergence of $\sum \frac{1}{n}$ implies the divergence of $\sum_{n=1}^{\infty} \frac{n}{n^2 + (-1)^n}$.

(3) Let $S$ and $T$ be nonempty subsets of $\mathbb{R}$ with the following property: $s < t$ for all $s \in S$ and $t \in T$. Then $\sup S < \inf T$.

True $\bigcirc$ False $\bigcirc$

Solution: False. The following is a counterexample. Let $S = (0, 1)$ and $T = (1, 2)$. Then $\sup S = 1 = \inf T$.

(4) If $f$ is uniformly continuous on $[k, \infty)$ for some $k$, then $f$ is bounded on $[k, \infty)$.

True $\bigcirc$ False $\bigcirc$

Solution: False. The following is a counterexample. $f(x) = x$ on $[k, \infty)$. $f$ is uniformly continuous by definition, but $f$ is not bounded on $[k, \infty)$.
Problem 2 (30 pt) Define a sequence \((x_n)\) by setting \(x_1 = 1\), and defining \(x_{n+1} = \sqrt{2x_n}\) for \(n \geq 1\).

(a) (10 pt) Is the sequence bounded? Justify your statement.

Solution: We prove \(0 < x_n < 2\) for all \(n\) by induction. When \(n = 1\) this holds. Assume it holds for \(n\), then for \(n + 1\), we have \(x_{n+1} = \sqrt{2x_n} > 0\), and \(\sqrt{2x_n} < \sqrt{2 \cdot 2} = 2\), so \(0 < x_{n+1} < 2\). Therefore \(0 < x_n < 2\) for all \(n\).

(b) (10 pt) Prove that \((x_n)\) is an increasing sequence.

Solution: Since from the previous question we have \(x_n > 0\) for all \(n\), so we only need to show \(x_{n+1} = \sqrt{2x_n} > x_n\) by showing \(\sqrt{2} > \sqrt{x_n}\), and since \(2 > x_n\) for all \(n\) (by the previous question) the inequality is true for the square root.

Another solution: we use induction to show \(x_{n+1} > x_n\) for all \(n\). When \(n = 1\), we have \(\sqrt{2} > 1\). Suppose it is true for \(n\), i.e. \(x_{n+1} > x_n\). Then \(2x_{n+1} > 2x_n > 0\) (the fact that \(x_n > 0\) is from part (a)), therefore \(\sqrt{2x_{n+1}} > \sqrt{2x_n}\), i.e. \(x_{n+2} > x_{n+1}\), so the statement also holds for \(n + 1\). This proves the statement for all \(n\).

(c) (10 pt) Does the sequence converge? If so, find \(\lim x_n\) and prove the convergence; if not, justify your answer.

Solution: since \(x_n\) is an increasing and bounded sequence, \(\lim x_n\) exists. Denote the limit by \(a\). Then take limits on both side of \(x_{n+1} = \sqrt{2x_n}\), we have \(a = \sqrt{2a}\) which gives \(a = 0\) or \(a = 2\). Since \((x_n)\) is increasing and \(x_1 = 1\), hence \(x_n \geq 1\) for all \(n\), therefore the limit cannot be 0. So \(\lim x_n = 2\).
Problem 3 (30 pt) Consider $f(x) = x^2(3 - x)$ and $g(x) = |f(x)|$, defined for all $x \in \mathbb{R}$.

(a) (10 pt) Use the $\epsilon - \delta$ definition of continuity to prove that $g$ is continuous at $x_0 = 3$.

Solution: Now we prove the continuity by definition. For any $\epsilon > 0$, let $\delta = \min\{1, \frac{1}{16}\}$, then for any $x$ with $|x - x_0| < \delta$, we have
\[
|g(x) - g(x_0)| = ||x^2(3 - x)| - 0| = |x^2(3 - x)| < 16\delta = \epsilon.
\]
Here we used that $|x - x_0| < \delta \leq 1$, therefore $2 < x < 4$ so $x^2 < 16$.

Remark: the choice of $\delta$ should only depend on $x_0$ and $\epsilon$, and should NOT depend on $x$ in the above expression. The reason is that $x$ is in a range determined by $\delta$ ($|x - x_0| < \delta$), if $\delta$ also depends on $x$, then it goes into an infinite circular argument that nothing will be fixed. Also $x$ is not a fixed number (it’s any number inside an interval), so $\delta$ cannot be fixed if its expression contains $x$.

(b) (10 pt) Prove that there are at least four solutions to the equation $g(x) = \frac{1}{2}$.

Solution: First note that for $x \leq 3$, $g(x) = x^2(3 - x)$ while for $x > 3$, $g(x) = x^2(x - 3)$. Since $f$ is a continuous function (as a product of continuous functions), therefore $g = |f|$ is also continuous. So we can apply intermediate function theorem.

Since $g(-1) = 4 > 1/2 > 0 = g(0)$, there exists $x_1 \in (-1, 0)$ such that $g(x_1) = 1/2$.

Similarly, since $g(0) = 0 < 1/2 < 2 = g(1)$, there exists $x_2 \in (0, 1)$ such that $g(x_2) = 1/2$.

Since $g_1 = 2 > 1/2 > 0 = g(3)$, so there exists $x_3 \in (1, 3)$ such that $g(x_3) = 1/2$.

And $g(3) = 0 < 1/2 < 16 = g(4)$, so there exists $x_4 \in (3, 4)$ such that $g(x_4) = 1/2$.

(c) (10 pt) Is $g$ uniformly continuous on $\mathbb{R}$? Justify your answer. Solution: No $g$ is not uniformly continuous on $\mathbb{R}$. We prove this by definition, i.e. we need to show: $\exists \epsilon_0 > 1$, such that $\forall \delta > 0$, there exists $x, y \in \mathbb{R}$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon_0$.

Let $\epsilon_0 = 1$, for any $\delta > 0$, let $n \in \mathbb{N}$ such that $n \geq 3$ and $\delta > 1/n$ (by Archimedean property there exists such $n$), and let $x = n$ and $y = n + \frac{1}{n}$. Then $|x - y| = \frac{1}{n} < \delta$.

And
\[
|f(x) - f(y)| = |n^2(n - 3) - (n + \frac{1}{n})^2(n + \frac{1}{n} - 3)|
\]
\[
= |(n^3 - 3n^2) - (n^3 + 3n + \frac{3}{n} - 3n^2 - 6 - \frac{3}{n^3})|
\]
\[
= |- (3n - 6 + \frac{3}{n} - \frac{3}{n^2} + \frac{1}{n^3})|
\]
\[
= |(3n - 6) + (\frac{3}{n} - \frac{3}{n^2} + \frac{1}{n^3})| \geq 3 > \epsilon_0
\]

Here we used $n \geq 3$ hence $3n - 6 \geq 3$, and $(\frac{3}{n} - \frac{3}{n^2}) > 0$, $\frac{1}{n^3} > 0$.

Therefore by definition $g$ is not uniformly continuous.

Remark: Having unbounded derivative on $\mathbb{R}$ is not a sufficient reason to show $f$ is not uniformly continuous. Recall the example of $\sqrt{x}$.
Problem 4 (20 pt) Consider the function $g$ on $[0, \infty)$ defined by

$$g(x) = \begin{cases} x^2(1-x) & x \in [0,1] \\ 0 & x > 1 \end{cases}$$

Define a sequence of functions $(f_n)$ on $[0,1]$ by $f_n(x) = g(nx)$.

(a) (10 pt) Find a function $f$ defined on $[0,1]$ such that $f_n \to f$ pointwise. Prove your assertion.

Solution: We can write $f_n$ as

$$f_n(x) = \begin{cases} (nx)^2(1-nx) & x \in [0, 1/n] \\ 0 & x \in (1/n, 1] \end{cases}$$

(The definition of $f_n$ depends on where $x$ is!)

Let $f(x) = 0$ on $[0,1]$. We show that $f_n \to f$ pointwise. For any $x \in [0,1]$, if $x = 0$ then $f_n(0) = 0$ for all $n$, hence $\lim_{n \to \infty} f_n(0) = 0 = f(0)$. Otherwise $1 \geq x > 0$, then there exists $N$ such that $1/N < x$. And for any $n > N$, we have $x > 1/N > 1/n$, hence $f_n(x) = 0$ for all $n > N$. Therefore by definition we have $\lim f_n(x) = 0$ for such $x$ as well.

(b) (10 pt) Does $f_n \to f$ uniformly on $[0,1]$? Prove your assertion.

Solution: $f_n$ does not converge uniformly to $f$. We use the following criterion:

$f_n \to f$ uniformly on $[0,1]$ if and only if $\lim_{n \to \infty} \sup_{x \in [0,1]} |f_n(x) - f(x)| = 0$.

Since in our case $f = 0$, we only need to compute the following sequence $\sup_{x \in [0,1]} |f_n(x)|$. Since $f_n(0) = 0$ and $f_n(x) = 0$ for $x \geq 1/n$, the maximum of $f_n$ is achieved at $x_n \in (0, 1/n)$ with $f'_n(x_n) = 0$. Compute $f'_n$ we get

$$f'_n(x) = 2n^2x(1-nx) - n^3x^2$$

and $f'_n(x_n) = 0$ gives $x_n = \frac{2}{3n}$ (the other critical point $x_n = 0$ is discarded since obviously it is not the maximum). Then $\sup_{x \in [0,1]} |f_n(x)| = f_n(x_n) = \frac{4}{27}$. (In fact, since $f_n$ is just a rescaling of $g$ in the $x$ direction, $\sup_{x \in [0,1]} |f_n(x)| = \sup_{x \in [0,n]} |g(x)| = \frac{4}{27}$ for all $n$.)

So we have $\lim_{n \to \infty} \sup |f_n(x)| = \frac{4}{27} \neq 0$, which implies $f_n$ does not converge uniformly to $f$. 
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