PARAMETRIZING HIGHER STRUCTURE MAPS FOR RESOLUTIONS OF LENGTH THREE

XIANGLONG NI

Abstract. We show how the action of the defect Lie algebra on Weyman's generic ring can be used to parametrize different choices of higher structure maps for a given free resolution of length three. Using the split exact case as a starting point, we then revisit the generators and relations of the generic ring from this perspective. In the process, we give an alternate treatment of the "generic top complex" for a split exact complex, unifying a number of constructions from previous work. We also obtain a geometric interpretation of the generic ring in relation to a certain generalized flag variety, and we find a close relationship between the generic rings for linked formats.

Contents

1. Introduction

Given a ring R and a finite free resolution

$$\mathbb{F}\colon 0 \to F_m \xrightarrow{d_m} \cdots \to F_1 \xrightarrow{d_1} F_0,$$

where $F_i = R^{f_i}$, we refer to the sequence $(f_0, f_1, \ldots, f_m)$ as the *format* of $\mathbb{F}$. In the event that R is local and $\mathbb{F}$ is minimal, the f_i are the ordinary Betti numbers of the module $H_0(\mathbb{F})$, but we will benefit from working in greater generality. Resolutions of format $(1, n, n-1)$ are described by the following theorem, first proven by Hilbert over polynomial rings and then later by Burch in generality:

Theorem (Hilbert-Burch)**.** *Let R_{univ} be the polynomial ring on the variables $x_{i,j}$ ($1 \le i \le n, 1 \le j \le n-1$) and another variable u. Consider the complex*

$$\mathbb{F}_{\text{univ}}\colon 0 \to R_{\text{univ}}^{n-1} \xrightarrow{d_2} R_{\text{univ}}^{n} \xrightarrow{d_1} R_{\text{univ}}$$

where $d_2 = [x_{i,j}]$ *is the generic matrix in the variables* $x_{i,j}$ *and* $(d_1)_{1,k}$ *is* $(-1)^k u$ *times the* k*-th* $(n-1) \times (n-1)$ *minor of* d_2*. Then*

(1) $\mathbb{F}_{\text{univ}}$ *is acyclic, and*
(2) *for any other resolution* $\mathbb{F}$ *of format* $(1, n, n-1)$ *over some ring* R*, there exists a (unique) map* $R_{\text{univ}} \to R$ *such that* $\mathbb{F} = \mathbb{F}_{\text{univ}} \otimes_{R_{\text{univ}}} R$*.*

Technically, the classical statement of Hilbert-Burch is only the latter point. The first claim moreso shows the robustness of the theorem: since $\mathbb{F}_{\text{univ}}$ is the universal example of a $(1, n, n-1)$ resolution, the Hilbert-Burch theorem is the *best possible* equational structure theorem for these resolutions.

The goal of finding structure theorems akin to Hilbert-Burch for other formats can thus be recast in terms of searching for universal resolutions. This was a project laid out by Hochster in [9]. However, Bruns [2] showed that for resolutions of length greater than two, requiring $R_{\text{univ}} \to R$ to be unique for a given $\mathbb{F}$ is too stringent. Relaxing this condition, he proved:

Theorem (Bruns [2, Theorem 1])**.** *Let* $r_1, \ldots, r_m$ *be nonnegative integers and let* $f_i = r_i + r_{i+1}$ *for* $i = 0, \ldots, m-1$ *and* $f_m = r_m$*. There exists a complex* $\mathbb{F}_{\text{gen}}$ *of format* $(f_0, \ldots, f_m)$ *over a ring* R_{gen} *such that*

(1) $\mathbb{F}_{\text{gen}}$ *is acyclic, and*
(2) *for any other resolution* $\mathbb{F}$ *of the same format over a ring* R*, there exists a homomorphism* $R_{\text{gen}} \to R$ *such that* $\mathbb{F} = \mathbb{F}_{\text{gen}} \otimes_{R_{\text{gen}}} R$*.*

We say that $(R_{\text{gen}}, \mathbb{F}_{\text{gen}})$ *is generic for the given format.*

While this settles the question of existence, the substance of e.g. Hilbert-Burch is not so much the existence of the universal example, but rather its explicit description. So if we hope to extract concrete structure theorems from the study of generic free resolutions, we should strive to explicitly understand the generators and relations of the generic rings.

Our case of interest will be resolutions of length three. For such resolutions over $\mathbb{C}$-algebras, Weyman constructed a generic pair $(\widehat{R}_{\text{gen}}, \mathbb{F}_{\text{gen}})$ in [16] for each format (f_0, f_1, f_2, f_3). Note that because the uniqueness condition was dropped in (2), there may be non-isomorphic generic rings for the same format, so we use the notation $\widehat{R}_{\text{gen}}$ to denote Weyman's construction specifically.

The acyclicity of Weyman's $\mathbb{F}_{\text{gen}}$ was not proven until much later, in [17]. The key insight was understanding the role played by a certain Kac-Moody Lie algebra $\mathfrak{g}$ in the construction of $\widehat{R}_{\text{gen}}$. Although we will not actually use the acyclicity of $\mathbb{F}_{\text{gen}}$ at any point in this paper, the Lie algebra $\mathfrak{g}$ will be essential for understanding $\widehat{R}_{\text{gen}}$ explicitly, which is one of our main goals.

This Lie algebra will be discussed in §3 in more detail, but for now we comment that $\widehat{R}_{\text{gen}}$ is Noetherian exactly when $\mathfrak{g}$ is finite-dimensional, i.e. of Dynkin type. The corresponding formats (f_0, f_1, f_2, f_3), enumerated below in Table 1, are called Dynkin formats. There are up to six per Dynkin type—fewer if the Dynkin diagram has symmetric arms.

In order to explain the structure of $\widehat{R}_{\text{gen}}$, we must first recall some basic results about Lie algebras in §2. Then in §3, we revisit the construction of $\widehat{R}_{\text{gen}}$, focusing on the action of the defect Lie algebra introduced in [16] which acts on $\widehat{R}_{\text{gen}}$ by derivations. We show that the exponential of this action can be used to relate different choices of maps $\widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}_{\text{gen}}$ to a fixed resolution $\mathbb{F}$ over R.

In particular, once one has a particular choice of $w \colon \widehat{R}_{\text{gen}} \to R$ for $\mathbb{F}$, this allows the parametrization of all choices in terms of w. In the language of existing literature such as [11], this gives formulas for the "generic higher structure maps" $v_j^{(i)}$ given a specific $w_j^{(i)}$.

TABLE 1. Length three formats with Noetherian $\widehat{R}_{\text{gen}}$

Type D_n	Type E_6	Type E_7	Type E_8	
$(1, n, n, 1)$	$(1, 5, 6, 2)$	$(1, 6, 7, 2)$	$(1, 7, 8, 2)$	Format I (dual to VI)
$(1, 4, n, n-3)$		$(1, 5, 7, 3)$	$(1, 5, 8, 4)$	Format II (linked to I)
$(n-3, n, 4, 1)$	$(2, 6, 5, 1)$	$(3, 7, 5, 1)$	$(4, 8, 5, 1)$	Format III (dual to II)
	$(2, 5, 5, 2)$	$(3, 6, 5, 2)$	$(4, 7, 5, 2)$	Format IV (linked to III)
		$(2, 5, 6, 3)$	$(2, 5, 7, 4)$	Format V (dual to IV)
		$(2, 7, 6, 1)$	$(2, 8, 7, 1)$	Format VI (linked to V)

Unfortunately, given an arbitrary resolution $\mathbb{F}$, we lack efficient ways of computing higher structure maps $w_j^{(i)}$ for $\mathbb{F}$ (equivalently, finding a homomorphism $\widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}_{\text{gen}}$ to $\mathbb{F}$). But in §4 we will show that this is quite easy in some special cases. This includes the case that $\mathbb{F}$ is a split exact complex.

The study of maps $\widehat{R}_{\text{gen}} \to R$ gives us one method of probing the generators and relations which define $\widehat{R}_{\text{gen}}$. In §6 we demonstrate how the observations from §3 and §4 allow us to restate a result from [17] describing the relations which hold in $\widehat{R}_{\text{gen}}$. For Dynkin formats, we find a subring of $\widehat{R}_{\text{gen}}$ whose spectrum is the affine cone over a generalized flag variety. We also observe that the generic rings for "linked formats" (see Table 1) share a subring.

In forthcoming papers, we will apply the techniques and results of this paper to study the linkage and structure theory of grade three perfect ideals $I \subset R$ with the property that R/I can be resolved by a resolution of Dynkin format. The main conjecture we intend to prove regarding linkage is the following, to which Theorem 6.10 is a precursor.

Conjecture 1.1 (Licci conjecture)**.** *Let I be a perfect ideal of grade 3 in a local Gorenstein $\mathbb{C}$-algebra R. Let $\mathbb{F}$ be a minimal free resolution of R/I. If the format of $\mathbb{F}$ is Dynkin, then I is in the linkage class of a complete intersection (licci).*

The Dynkin assumption is necessary, as perfect ideals of grade three are otherwise not necessarily licci. A simple example is $I = (x, y, z)^2 \subset R = \mathbb{C}[x, y, z]_{(x,y,z)}$. The resolution of R/I in this case has format $(1, 6, 8, 3)$ of type $\widetilde{E}_7$ and the ideal I is perfect but not licci. In fact Conjecture 1.1 is sharp: in [7, Theorem 3.2] it is shown that for every non-Dynkin format, there exists a non-licci perfect ideal of grade 3 in $\mathbb{C}[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ whose minimal free resolution has that format.

We also intend to prove the following conjecture providing a finite family of generic examples for perfect ideals with Dynkin resolutions. It is a refinement of the "Genericity conjecture" appearing in places such as [18]. Given a Dynkin format $(1, f_1, f_2, f_3)$, let $X^w \subset G/P_{x_1}$ be the codimension three Schubert variety defined in Theorem 6.8, and let $[v] \in G/P_{x_1}$ be the B_+-fixed point, i.e. the highest weight line in the Plücker embedding in $\mathbb{P}(V(\omega_{x_1}))$.

For each $[\sigma] \in W_{P_{z_1}} \backslash W / W_{P_{x_1}}$ with $[\sigma] \neq [e]$ (where $e \in W$ is the identity), we take a representative $\sigma \in W$ of $[\sigma]$ and consider the point $\sigma \cdot [v] \in G/P_{x_1}$. The local ring $S_{[\sigma]} = \mathcal{O}_{G/P_{x_1}, \sigma\cdot[v]}$ is isomorphic to a polynomial ring localized at the ideal of variables. The point $\sigma \cdot [v]$ also lies in X^w; let $I_{[\sigma]} \subset S_{[\sigma]}$ be the defining ideal of X^w at this point.

Conjecture 1.2 (Local genericity conjecture)**.** *Suppose I is a perfect ideal of grade 3 in a local $\mathbb{C}$-algebra R such that R/I has a (not necessarily minimal) resolution of Dynkin format $(1, f_1, f_2, f_3)$. Then there exists a unique $[\sigma] \in W_{P_{z_1}} \backslash W / W_{P_{x_1}}$, $[\sigma] \neq [e]$, such that there exists a local homomorphism*

$\varphi\colon S_{[\sigma]} \to R$ such that $\varphi(I_{[\sigma]})R = I$. Note that since $I_{[\sigma]}$ and I are both perfect ideals of grade three, the resolution of $S_{[\sigma]}/I_{[\sigma]}$ specializes to one for R/I via φ.

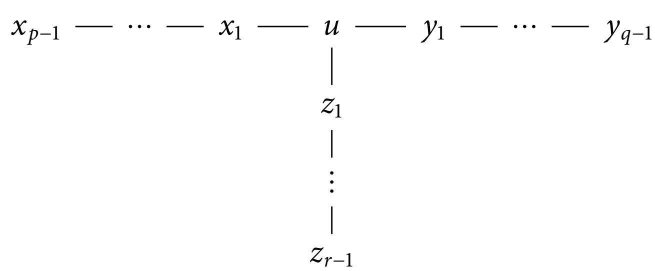
These Schubert varieties X^w were studied in [15], with the expectation that they would be closely related to $\widehat{R}_{\text{gen}}$. There is indeed a deep connection between them, which we will explain in Theorem 6.8. In [13], the resolutions for X^w restricted to the opposite big open cell were constructed. Localizing at the "origin" of this affine patch, which is to say the B_--fixed point in G/P_{x_1}, one obtains the resolution for $S_{[w_0]}/I_{[w_0]}$ where $w_0 \in W$ is the longest element.

The author is deeply thankful to David Eisenbud for introducing him to this topic, and to Jerzy Weyman and Lorenzo Guerrieri for fruitful discussions which were essential in motivating many of the results in this paper.

2. Background on Lie algebras

We first summarize some basic results on Lie algebras that will be needed throughout the rest of the paper. Sections §3 and §4 mostly do not depend on the material here, so the reader may skip ahead and refer back as needed. However, §6 will use Lie algebras and representation theory more heavily.

Fix positive integers p, q, r and let $T = T_{p,q,r}$ denote the graph



Let $n = p + q + r - 2$ be the number of nodes. From the above graph, we construct an $n \times n$ matrix A, called the *Cartan matrix*, whose rows and columns are indexed by the nodes of T:

$$A = (a_{i,j})_{i,j\in T}, \qquad a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i, j \in T \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

2.1. **Construction of the Kac-Moody Lie algebra.** The matrix A defined above has rank n unless the graph T is one of the affine Dynkin diagrams $\widetilde{E}_n$. For simplicity we will assume this is not the case. We are primarily interested in when T is one of the (ordinary) Dynkin diagrams D_n or E_n.

Let $\mathfrak{h} = \mathbb{C}^n$, and take $\Pi = \{\alpha_i\}_{i\in T}$ in $\mathfrak{h}^*$ to be the coordinate functions. These are the *simple roots*. Let $\Pi^\vee = \{\alpha_i^\vee\}_{i\in T}$ be elements of $\mathfrak{h}$ such that

$$\langle \alpha_i^\vee, \alpha_j \rangle = a_{i,j}.$$

These are the *simple coroots*. The Kac-Moody Lie algebra $\mathfrak{g}(T)$ is generated by elements e_i, f_i for $i \in T$, subject to the defining relations

$$[e_i, f_j] = \delta_{i,j}\alpha_i^\vee,$$
$$[h, e_i] = \langle h, \alpha_i\rangle e_i, [h, f_i] = -\langle h, \alpha_i\rangle f_i \text{ for } h \in \mathfrak{h},$$
$$[h, h'] = 0 \text{ for } h, h' \in \mathfrak{h},$$
$$\operatorname{ad}(e_i)^{1-a_{i,j}}(e_j) = \operatorname{ad}(f_i)^{1-a_{i,j}}(f_j) \text{ for } i \neq j.$$

For brevity, we will often just write $\mathfrak{g}$ for $\mathfrak{g}(T)$. Under the adjoint action of $\mathfrak{h}$, the Lie algebra $\mathfrak{g}$ decomposes into eigenspaces as $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$, where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

This is the *root space decomposition* of $\mathfrak{g}$. Let $Q \subset \mathfrak{h}^*$ be the lattice $\bigoplus_{i \in T} \mathbb{Z}\alpha_i$. If $\mathfrak{g}_\alpha \neq 0$, then necessarily $\alpha \in Q$. If such an α is nonzero, we say it is a *root*. We note that the Lie algebra $\mathfrak{g}$ is Q-graded. Using the homomorphism $Q \to \mathbb{Z}$ sending each $\alpha_i \mapsto 1$, we can coarsen this to a $\mathbb{Z}$-grading, called the *principal gradation* on $\mathfrak{g}$. In this grading, $\mathfrak{g}_0 = \mathfrak{h}$, and we let $\mathfrak{g}_+, \mathfrak{g}_-$ denote the positive and negative parts respectively. We write $\mathfrak{b}_+, \mathfrak{b}_-$ for the nonnegative and nonpositive parts (i.e. $\mathfrak{b}_+ = \mathfrak{h} + \mathfrak{g}_+$); these are *(opposite) Borel subalgebras*.

2.2. **Representations.** Let V be a representation of $\mathfrak{g}$, or equivalently a $\mathfrak{g}$-module. For $\lambda \in \mathfrak{h}$, define the λ*-weight space of* V to be

$$V_\lambda = \{v \in V : h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\lambda \neq 0$, then we say λ is a *weight* of V. A nonzero vector $v \in V_\lambda$ is a *highest weight vector* if $\mathfrak{g}_+ \cdot v = 0$. If such a v generates V as a $\mathfrak{g}$-module, then we say V is a *highest weight module* with highest weight λ.

Let $\mathcal{U}$ denote the *universal enveloping algebra* functor. Given $\lambda \in \mathfrak{h}^*$, the *Verma module* $M(\lambda)$ is defined to be

$$M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_+)} \mathbb{C}_\lambda.$$

Here $\mathbb{C}_\lambda$ is the $\mathfrak{b}_+$-module where $\mathfrak{h}$ acts by λ and $\mathfrak{g}_+$ acts trivially. All the weights of $M(\lambda)$ are in $\lambda + Q$. If $v \in V_\lambda$ is a highest weight vector, then there is a map $M(\lambda) \to V$ sending $1 \mapsto v$. If V is a highest weight module then this map is surjective.

Every Verma module $M(\lambda)$ has a unique maximal proper submodule $J(\lambda)$. It follows that $V(\lambda) = M(\lambda)/J(\lambda)$ is an irreducible highest weight module with heighest weight λ, and any such module is isomorphic to $V(\lambda)$.

Let $\omega_i \in \mathfrak{h}^*$ be the basis dual to $\alpha_i^\vee$. Explicitly, ω_i is the linear combination of α_i given by the i-th column of A^{-1}. These are the *fundamental weights*, and the representations $V(\omega_i)$ are called *fundamental representations*.

One can alternatively work with lowest weights instead of highest weights, replacing $\mathfrak{g}_+$, $\mathfrak{b}_+$ by $\mathfrak{g}_-$, $\mathfrak{b}_-$ in all of the preceding.

2.3. **The grading induced by a node of $T_{p,q,r}$.** As mentioned previously, the Lie algebra $\mathfrak{g}$ is Q-graded, where $Q = \bigoplus_{i \in T} \mathbb{Z}\alpha_i$. Let $I \subseteq T$. Consider the group homomorphism

$$\bigoplus_{i \in T} \mathbb{Z}\alpha_i \to \bigoplus_{i \in I} \mathbb{Z}\alpha_i$$

which sends α_i to itself if $i \in I$ and zero otherwise. This coarsens the $Q \cong \mathbb{Z}^n$-grading on $\mathfrak{g}$ to a $\mathbb{Z}^{|I|}$-grading.

We will be interested in the case that $I = \{t\}$ is a singleton set, so the result is a $\mathbb{Z}$-grading on $\mathfrak{g}$, which we will call the *t-grading*. Let $h_i \in \mathfrak{h}$ be the basis dual to α_i. Explicitly, h_i is the linear combination of $\alpha_i^\vee$ given by the i-th column of A^{-1}. The $\mathbb{Z}$-grading induced by $t \in T$ is the decomposition of $\mathfrak{g}$ into eigenspaces for the adjoint action of h_t:

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \ker(\operatorname{ad}(h_t) - k).$$

The $k = 0$ component has the form

$$\ker \operatorname{ad}(h_t) = \mathfrak{g}^{(t)} \oplus \mathbb{C}h_t$$

where $\mathfrak{g}^{(t)}$ is the Lie algebra generated by $\{e_i, f_i\}_{i\neq t}$, i.e. the Lie algebra corresponding to the diagram $T - \{t\}$.

Let $v \in V(\lambda)$ be a highest weight vector, and let $a_t = \langle h_t, \lambda\rangle$. Then $h_t \cdot v = a_t v$, and the eigenvalues for the action of h_t on $V(\lambda)$ are $a_t, a_t-1, \ldots$. This list terminates iff $V(\lambda)$ is finite-dimensional. Each eigenspace is a finite-dimensional representation of the subalgebra $\mathfrak{g}^{(t)} \times \mathbb{C}h_t$. In particular, v is a highest weight vector for the eigenspace with value a_t, thus this top component is the fundamental representation of $\mathfrak{g}^{(t)}$ with highest weight $\sum_{i\neq t} c_i\omega_i$ if $\lambda = \sum_{i\in T} c_i\omega_i$.

Example 2.1. Consider $T_{2,3,3} = E_6$, and let $t = z_1$. The diagram $E_6 - \{z_1\}$ consists of the A_4 diagram y_2, y_1, u, x_1 and the A_1 diagram z_2. So if we let $F = \mathbb{C}^5$ and $F' = \mathbb{C}^2$, the subalgebra $\mathfrak{g}^{(t)}$ is $\mathfrak{sl}(F) \times \mathfrak{sl}(F')$.

The decomposition of $\mathfrak{g}$ is

$$\begin{aligned}
\ker(\mathrm{ad}(h_t) - 2) &= \bigwedge^4 F \otimes \bigwedge^2 F'^* \\
\ker(\mathrm{ad}(h_t) - 1) &= \bigwedge^2 F \otimes F'^* \\
\ker(\mathrm{ad}(h_t) - 0) &= \mathfrak{sl}(F) \oplus \mathfrak{sl}(F') \oplus \mathbb{C}h_t \\
\ker(\mathrm{ad}(h_t) + 1) &= \bigwedge^2 F^* \otimes F' \\
\ker(\mathrm{ad}(h_t) + 2) &= \bigwedge^4 F^* \otimes \bigwedge^2 F'
\end{aligned}$$

As another example, consider the fundamental representation $V(\omega_{z_2})$. The coefficient of α_t in ω_{z_2} is 5/3, and the representation decomposes into eigenspaces for h_t as

$$\begin{aligned}
\ker(h_t - 5/3) &= F' \\
\ker(h_t - 2/3) &= \bigwedge^3 F \\
\ker(h_t + 1/3) &= F' \otimes F \\
\ker(h_t + 4/3) &= F^*.
\end{aligned}$$

By slight abuse of notation, here h_t refers to the action of h_t on the representation.

3. Parametrizing higher structure maps

All rings considered throughout this paper are $\mathbb{C}$-algebras. Fix a format (f_0, f_1, f_2, f_3) and let $r_i = \sum_{j=i}^{3}(-1)^{j-i} f_j$ denote the rank of the differential d_i in any resolution of the given format. For each such format, Weyman constructed the pair $(\widehat{R}_{\mathrm{gen}}, \mathbb{F}^{\mathrm{gen}})$ in [16] and proved its genericity in [17], meaning that for any resolution $(R, \mathbb{F})$ of the given format, there exists a map $w\colon \widehat{R}_{\mathrm{gen}} \to R$ for which $\mathbb{F} = \mathbb{F}^{\mathrm{gen}} \otimes R$.

The map w is *not* uniquely determined by the resolution $\mathbb{F}$. This is an inevitable feature of the construction, as Bruns showed this is necessarily the case for any generic ring for formats of length at least three [2, Theorem 2]. As such, it is natural to ask how different choices of w for the same $\mathbb{F}$ are related to one another. For this, we need to briefly recall the construction of Weyman's $\widehat{R}_{\mathrm{gen}}$. We refer the reader to [16, §2] for details.

One starts with the Buchsbaum-Eisenbud multiplier ring R_a. A resolution $(R, \mathbb{F})$ determines a unique map $R_a \to R$; this is just the statement of the First Structure Theorem in [4]. The ring R_a carries a complex $\mathbb{F}^a$ which is the generic example of a complex acyclic in codimension one.

To obtain $\mathbb{F}^{\text{gen}}$ from $\mathbb{F}^a$, we need to increase the depth of $I(d_3)$ from two to three. This is achieved by killing H^2 in the Koszul complex

$$0 \to \bigwedge^0 \mathcal{K} \to \bigwedge^1 \mathcal{K} \to \bigwedge^2 \mathcal{K} \to \bigwedge^3 \mathcal{K}$$

where $\mathcal{K} = \bigwedge^{r_3} F_3^* \otimes \bigwedge^{r_3} F_2$. This process is performed inductively using the defect Lie algebra $\mathbb{L}$, by adjoining variables for the coordinates of p_i according to the diagram below:

$$\begin{array}{ccccccc} 0 \longrightarrow & \bigwedge^0 \mathcal{K} & \longrightarrow & \bigwedge^1 \mathcal{K} & \longrightarrow & \bigwedge^2 \mathcal{K} & \longrightarrow \bigwedge^3 \mathcal{K} \\ & & & \uparrow{\scriptstyle p_i} & & \uparrow{\scriptstyle q_i} & \\ & & & \mathbb{L}_i^* & \longrightarrow & (\bigwedge^2 \mathbb{L}^*)_i & \end{array} \tag{3.1}$$

(Here $\mathbb{L}$ really denotes the tensor product of $\mathbb{L}$ with the ring already constructed.) The lower horizontal map is dual to the bracket in $\mathbb{L}$ and the map q_i is defined from lower p_i. The map q_1 is defined using the Second Structure Theorem of [4].

After adjoining the coordinates of $p_1, \dots, p_m$, quotienting by appropriate relations, and taking an ideal transform, one obtains the ring R_m. The ring $\widehat{R}_{\text{gen}}$ is then defined to be the direct limit of the rings R_m.

We have skimmed over a lot of details, but the important point is that a resolution $(R, \mathbb{F})$ together with a choice of structure maps p_i for $\mathbb{F}$ in accordance with the diagram above determines a map $\widehat{R}_{\text{gen}} \to R$. Here one sees the non-uniqueness mentioned before: after having computed $p_1, \dots, p_{i-1}$, there is a $\text{Hom}(\mathbb{L}_i^*, R)$ of choices for the map p_i. In other words, $\mathbb{L}_i$ records the failure of the map p_i to be uniquely determined—hence the name "defect" Lie algebra.

In [16], it is shown how elements $u \in \mathbb{L}_n$ act on $\widehat{R}_{\text{gen}}$ by R_{n-1}-linear derivations[1]. It is sufficient to describe how they affect (the coordinates of) p_{n+k} for $k \geq 0$, and this is as follows: the derivation D_u sends p_n^* to

$$\mathcal{K}^* \xrightarrow{\bigwedge^{r_3} d_3^*} \widehat{R}_{\text{gen}} \xrightarrow{u} \mathbb{L}_n \otimes \widehat{R}_{\text{gen}}$$

and p_{n+k}^* to

$$\mathcal{K}^* \xrightarrow{p_k^*} \mathbb{L}_k \otimes \widehat{R}_{\text{gen}} \xrightarrow{[u,-]} \mathbb{L}_{n+k} \otimes \widehat{R}_{\text{gen}}.$$

These are just restatements of the formulas given in [16, Prop. 2.11] and [16, Thm. 2.12] respectively.

These formulas naturally extend to an arbitrary element $X \in \mathbb{L} = \prod_{i>0} \mathbb{L}_i$; the resulting derivation is well-defined because $\mathbb{L}_{>n}$ acts by zero on R_n. In a slight abuse of notation, we will also write X for the corresponding derivation. Homomorphisms $\widehat{R}_{\text{gen}} \to R$ correspond to R-algebra homomorphisms $\widehat{R}_{\text{gen}} \otimes R \to R$, and the Lie algebra $\mathbb{L} \otimes R$ acts on $\widehat{R}_{\text{gen}} \otimes R$.

For $X \in \mathbb{L} \otimes R$, the action of $\exp X \coloneqq \sum_{i \geq 0} \frac{1}{i!} X^i$ on $\widehat{R}_{\text{gen}} \otimes R$ is well-defined since every element of $\widehat{R}_{\text{gen}} \otimes R$ is killed by a sufficiently high power of X. Since X acts by an $(R_a \otimes R)$-linear derivation, it follows formally that $\exp X$ acts by an automorphism fixing $R_a \otimes R$. Such automorphisms completely describe the non-uniqueness of the map $\widehat{R}_{\text{gen}} \to R$ given a particular resolution $(R, \mathbb{F})$, as the following result shows.

Theorem 3.1. *Let $\mathbb{F}$ be a resolution of length three over R and let $\widehat{R}_{\text{gen}}$ be the generic ring for the associated format. Fix a $\mathbb{C}$-algebra homomorphism $w\colon \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}$. Then w determines*

[1]Note that we write $\mathbb{L}_i$ here for $\mathbf{q}_i$ in that paper.

a bijection

$$\mathbb{L} \otimes R \simeq \{\mathbb{C}\text{-algebra homomorphisms } w' \colon \widehat{R}_{\text{gen}} \to R \text{ specializing } \mathbb{F}^{\text{gen}} \text{ to } \mathbb{F}\}.$$

Note that a $\mathbb{C}$*-algebra homomorphism* $\widehat{R}_{\text{gen}} \to R$ *can be viewed as an* R*-algebra homomorphism* $R \otimes \widehat{R}_{\text{gen}} \to R$*. The correspondence above identifies* $X \in \mathbb{L} \otimes R$ *with the map* $w \exp X$ *obtained by precomposing* w *with the action of* $\exp X$ *on* $R \otimes \widehat{R}_{\text{gen}}$*.*

Proof. The homomorphism $w \colon \widehat{R}_{\text{gen}} \otimes R \to R$ is completely determined by the choice of the structure maps p_i. For $X \in \mathbb{L} \otimes R$, let us write $X = \sum_{i>0} u_i$ where $u_i \in \mathbb{L}_i$, and let $X_n = \sum_{i=1}^n u_i$ denote the partial sums.

Precomposing w by $\exp X$ or $\exp X_n$ has the same effect on the structure maps p_k for $k \le n$. Acting by $\exp X$ on p_1, we get

$$p_1 + (\bigwedge^{r_3} d_3) u_1^*.$$

Here u_1^* means the dual of $R \xrightarrow{u_1} \mathbb{L} \otimes R$. All possible choices of the structure map p_1 are obtained by lifting a particular map q_1 in the diagram (3.1), so it follows that choices of $u_1 \in \mathbb{L}_1$ correspond to choices for the structure map p_1.

In general, acting by $\exp X$ on p_n gives

$$(p_n + p_{n-1}[u_1, -]^* + \cdots) + (\bigwedge^{r_3} d_3) u_n^*.$$

The first part consists of terms involving u_k for $k < n$. Once again, (3.1) shows that choices of $u_n \in \mathbb{L}_n$ correspond to choices for the structure map p_n. Proceeding inductively in this fashion, we get the desired statement. □

In the sequel we will not be so concerned with the structure maps p_i—rather, we will apply Theorem 3.1 to study a different family of structure maps that exist in $\widehat{R}_{\text{gen}}$. Let $p = f_0 + 1$, $q = f_1 - f_0 - 1$, and $r = f_3 + 1$, and let $\mathfrak{g} = \mathfrak{g}(T_{p,q,r})$ be the Kac-Moody Lie algebra associated to the graph $T_{p,q,r}$, as defined in §2. Let $F_i = \mathbb{C}^{f_i}$. The graph $T - \{z_1\}$ consists of

$$y_{q-1} \,—\, \cdots \,—\, y_1 \,—\, u \,—\, x_1 \,—\, \cdots \,—\, x_{p-1},$$

which we take to be $\mathfrak{sl}(F_1)$, and

$$z_2 \,—\, \cdots \,—\, z_{r-1}$$

which we take to be $\mathfrak{sl}(F_3)$. Hence in the z_1-grading on $\mathfrak{g}$ and its representations, each component is a representation of $\mathfrak{sl}(F_3) \times \mathfrak{sl}(F_1) \times \mathbb{C}h_{z_1}$, c.f. §2.3.

The connection to the preceding theory is that, with this setup, the defect Lie algebra $\mathbb{L}$ is exactly the negative part of $\mathfrak{g}$ and the previously described action of $\mathbb{L}$ on $\widehat{R}_{\text{gen}}$ extends to an action of $\mathfrak{g}$. This was instrumental in proving the acyclicity of $\mathbb{F}^{\text{gen}}$ in [17].

Example 3.2. For the format $(1, 5, 6, 2)$, the graph $T_{p,q,r}$ is $T_{2,3,3} = E_6$. The defect Lie algebra has two graded components: $F_3 \otimes \bigwedge^2 F_1^*$ and $\bigwedge^2 F_3 \otimes \bigwedge^4 F_1^*$. This is the negative part of $\mathfrak{g}$ as written out in Example 2.1.

Inside of the ring $\widehat{R}_{\text{gen}}$, there exist three representations of $\mathfrak{sl}(F_2) \times \mathfrak{sl}(F_0) \times \mathfrak{g}(T_{p,q,r})$ of particular interest, namely those generated by the entries of the differentials d_i. We call them the *critical* *representations*; they are

$$W(d_3) = F_2^* \otimes V_-(-\omega_{z_{r-1}}) = F_2^* \otimes [F_3 \oplus \bigwedge^{r_0+1} F_1 \oplus \cdots]$$
$$W(d_2) = F_2 \otimes V_-(-\omega_{y_{q-1}}) = F_2 \otimes [F_1^* \oplus F_3^* \otimes \bigwedge^{r_0} F_1 \oplus \cdots]$$
$$W(d_1) = F_0^* \otimes V_-(-\omega_{x_{p-1}}) = F_0^* \otimes [F_1 \oplus F_3^* \otimes \bigwedge^{r_0+2} F_1 \oplus \cdots]$$

Here $V_-(-\lambda)$ denotes the fundamental representation of $\mathfrak{g}$ with lowest weight $-\lambda$, and similarly for the others. Note that if this representation is finite-dimensional, then $V_-(-\lambda)$ and $V(\lambda)$ are dual to one another; in general one needs to take the graded dual instead. The bottom two components of each representation in the z_1-grading have been indicated above.

Given a map $w\colon \widehat{R}_{\text{gen}} \to R$ for a complex $(R, \mathbb{F})$, we denote by $w^{(i)}$ the restriction of w to the representation $W(d_i) \subset R_{\text{gen}}$, and by $w_j^{(i)}$ the restriction to the j-th graded component of that representation, counted from the bottom—so for instance $w_0^{(i)}$ is[2] the differential d_i. We call the maps $w_{>0}^{(i)}$ (a specific choice of) *higher structure maps* for $\mathbb{F}$. Theorem 3.1 shows explicitly how $\mathbb{L} \otimes R$ parametrizes choices of such maps.

Example 3.3. Consider a free resolution $\mathbb{F}$ of format $(1, f_1, f_2, f_3)$ resolving R/I where depth $I \geq 2$. The structure maps $w_1^{(i)}$ give a choice of multiplicative structure on $\mathbb{F}$; see [11, Prop. 7.1]. Explicitly, such a resolution has the (non-unique) structure of a commutative differential graded algebra, and the non-uniqueness is evidently seen from the fact that the multiplication $\bigwedge^2 F_1 \to F_2$ may be chosen as any lift in the diagram

(3.2)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_3 & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & R \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & \bigwedge^2 F_1 & & \end{array}$$

where the map $\bigwedge^2 F_1 \to F_1$ is given by $e_1 \wedge e_2 \mapsto d_1(e_1)e_2 - d_1(e_2)e_1$. Indeed, we have that $\mathbb{L}_1 = F_3 \otimes \bigwedge^2 F_1^*$, which is exactly the non-uniqueness witnessed here.

Now suppose that $w\colon R_{\text{gen}} \to R$ (equivalently, $R \otimes R_{\text{gen}} \to R$) is one choice of higher structure maps for $\mathbb{F}$, and take an element $X = \sum_{i>0} u_i \in \mathbb{L} \otimes R$ using the same notation as before. Let $w' = w \exp(X)$, i.e.

$$w' = w\left(1 + u_1 + \left(\frac{1}{2}u_1^2 + u_2\right) + \cdots\right)$$

Note that u_k maps $W(d_i)_j$ to $W(d_i)_{j-k}$. If we restrict the above equation to the representation $W(d_3)$ and expand it degree-wise, we get

$$w_0'^{(3)} = w_0^{(3)}$$
$$w_1'^{(3)} = w_1^{(3)} + w_0^{(3)} u_1$$
$$w_2'^{(3)} = w_2^{(3)} + w_1^{(3)} u_1 + w_0^{(3)} \left(\frac{1}{2}u_1^2 + u_2\right)$$
$$\vdots$$

[2]Technically this definition gives a map $w_0^{(i)}\colon F_i \otimes F_{i-1}^* \to R$ rather than $F_i \to F_{i-1}$, but this is evidently the same data—we will often abuse notation in this manner and implicitly adjust the source/target of maps as is convenient.

The first equation reflects that the underlying complex is still the same $\mathbb{F}$. The next equation shows that the new multiplication, viewed as a map $F_2^* \otimes \bigwedge^2 F_1 \to R$, was obtained from the old one by adding the composite

$$F_2^* \otimes \bigwedge^2 F_1 \xrightarrow{1\otimes u_1} F_2^* \otimes F_3 \xrightarrow{d_3} R.$$

Here $u_1 \in \mathbb{L}_1 = F_3 \otimes \bigwedge^2 F_1^*$ could've been any map $\bigwedge^2 F_1 \to F_3$, and this exactly matches what we see in (3.2).

4. The graded setting

We have shown how, starting from a particular choice of higher structure maps $w\colon \widehat{R}_{\text{gen}} \to R$ for a resolution $(R, \mathbb{F})$, it is possible to obtain all other choices of w using the exponential action of $\mathbb{L} \otimes R$. However, we still lack satisfactory methods for computing a specific w in the first place. This is addressed for the type D_n formats $(1, n, n, 1)$ and $(1, 4, n, n-3)$ in [12]. For arbitrary formats, a couple of structure maps $w_j^{(i)}$ for small j are treated in [8], but as the reader can see there, if one wants to do this for arbitrary formats, the situation rapidly balloons in difficulty as j increases.

Note that the structure maps p_i which determine the map $\widehat{R}_{\text{gen}} \to R$ are defined inductively via lifting. Thus if the resolution $\mathbb{F}$ is graded with differentials homogeneous of degree zero, then it is possible to choose maps p_i respecting this grading as well. To precisely leverage this, it is necessary to state the components of $\mathbb{L}$ and the higher structure maps $GL(F_i)$-equivariantly and not just $SL(F_i)$-equivariantly. Let $M_i = \bigwedge^{f_i} F_i$ and $M = M_1 \otimes M_2^* \otimes M_3$. In [16], we see that p_1 is really a map

$$\bigwedge^{r_1+1} F_1 \otimes F_3^* \otimes M^* \to \mathcal{K} = \bigwedge^{r_3} F_3^* \otimes \bigwedge^{r_3} F_2.$$

In particular $\mathbb{L}_1 \cong \bigwedge^{r_1+1} F_1^* \otimes F_3 \otimes M$ as a representation of $\prod GL(F_i)$.

Remark 4.1. The Buchsbaum-Eisenbud multiplier a_1 is an injective map $M \to M_0$. In the event that $\mathbb{F}$ resolves R/I where depth $I \geq 2$, a_1 yields an isomorphism $M \cong R$. This setting is the primary one of interest, and the identification $M \cong R$ has been implicitly made in much of the existing literature. This is the case for the decompositions of critical representations tabulated in [11]; to have these descriptions hold more generally, the jth graded component should be tensored with $M^{\otimes(-j)}$.

Although the observation that we can pick graded structure maps is elementary, it offers great mileage.

Example 4.2. Suppose that $\mathbb{F}$ is a graded resolution over a nonnegatively graded ring R. If all generators of $\mathbb{L}_1$ are in positive degrees, then there is a unique choice of graded higher structure maps, and moreover these structure maps $w_j^{(i)}, p_j$ are zero for $j \gg 0$ by degree considerations.

Example 4.3. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ with $\deg(x_i) > 0$, and let $I \subset R$ be a grade three perfect ideal. If

$$0 \to \bigoplus R(-b_{3j}) \to \bigoplus R(-b_{2j}) \to \bigoplus R(-b_{1j}) \to R$$

is a graded minimal free resolution of I, and

$$\max\{b_{3j}\} \leq 2\min\{b_{1j}\} \tag{4.1}$$

then $\mathbb{L}_1$ is generated in negative degrees, and the entries of all higher structure maps will have strictly positive degree—in particular, all structure maps will be zero mod $(x_1, \ldots, x_n)$. By [10], the condition (4.1) also implies I is not licci, hinting at a connection between higher structure maps $w_j^{(i)}$ and linkage. This is studied explicitly in [8] for small values of j and will be generalized in a forthcoming paper.

We will use Example 4.2 in the following special case. Suppose that R is an arbitrary (ungraded) $\mathbb{C}$-algebra but that the differential d_3 of $\mathbb{F}$ is a split inclusion. After choosing a splitting, $\mathbb{F}$ can be written as

$$0 \to F_3 \xrightarrow{\begin{bmatrix} I \\ 0 \end{bmatrix}} F_3 \oplus C \xrightarrow{\begin{bmatrix} 0 & d_2 \end{bmatrix}} F_1 \to F_0.$$

We can view this as a graded resolution where R, C, F_1, F_0 are entirely concentrated in degree 0, but F_3 is in degree 1.

Lemma 4.4. *Suppose that d_3 of $\mathbb{F}$ is a split inclusion. Then there is a choice of higher structure maps $w^{(i)}_{>0}$ for $\mathbb{F}$ in which only $w^{(3)}_1$ and $w^{(2)}_1$ are nonzero.*

Proof. We view $\mathbb{F}$ with the grading described above and choose structure maps that are homogeneous of degree zero. These will have the desired property as we can see from degree considerations: $W(d_1)_j$ is concentrated in degree $-j$, while $W(d_2)_j$ and $W(d_3)_j$ are concentrated in degrees $-j$ and $-j+1$ since $F_2 = C \oplus F_3$ is in degrees 0 and 1. □

In [11, Prop. 7.1] it is shown how the structure maps $w^{(3)}_1$ and $w^{(2)}_1$ can be computed via a comparison map from a Buchsbaum-Rim complex. We restate the lifting explicitly here, both for the sake of completeness and also to clarify the role played by M, since the identification $M \cong R$ was implicitly made in [11]. The First Structure Theorem in [4] gives a factorization

$$\begin{array}{ccc} \bigwedge^{r_1} F_1 & \xrightarrow{\bigwedge^{r_1} d_1} & \bigwedge^{r_1} F_0 \\ & \searrow \quad M \quad \nearrow_{a_1} & \end{array}$$

in particular a map $\beta\colon M^* \otimes \bigwedge^{r_1} F_1 \to R$, which is essentially a_2^* after appropriate identifications. It is straightforward to check that the composite

$$M^* \otimes \bigwedge^{r_1+1} F_1 \to M^* \otimes \bigwedge^{r_1} F_1 \otimes F_1 \xrightarrow{\beta\otimes 1} F_1 \xrightarrow{d_1} F_0$$

is zero, thus we can lift through d_2 to obtain a map

$$w^{(3)}_1\colon M^* \otimes \bigwedge^{r_1+1} F_1 \to F_2.$$

The difference of the two maps

$$M^* \otimes \bigwedge^{r_1} F_1 \otimes F_2 \xrightarrow{\beta\otimes 1} F_2$$

$$M^* \otimes \bigwedge^{r_1} F_1 \otimes F_2 \xrightarrow{1\otimes d_2} M^* \otimes \bigwedge^{r_1} F_1 \otimes F_1 \to M^* \otimes \bigwedge^{r_1+1} F_1 \xrightarrow{w^{(3)}_1} F_2$$

has image landing in $\ker d_2$, and thus it can be lifted through d_3 to obtain

$$w^{(2)}_1\colon M^* \otimes \bigwedge^{r_1} F_1 \otimes F_2 \to F_3.$$

In the case that $r_0 = 1$, these maps can be viewed as giving a choice of multiplication on the resolution

$$0 \to M^* \otimes F_3 \to M^* \otimes F_2 \to M^* \otimes F_1 \xrightarrow{\beta} R.$$

We conclude this section by showing that there is a particularly simple choice of higher structure maps for a split exact complex.

Let V_1, V_2, V_3 be the representations of $\mathfrak{g}(T_{p,q,r})$ with lowest weights $-\omega_{x_{p-1}}, -\omega_{y_{q-1}}, -\omega_{z_{r-1}}$ respectively, so that the critical representations are $W(d_3) = F_2^* \otimes V_3$, $W(d_2) = F_2 \otimes V_2$, and $W(d_1) = F_0^* \otimes V_1$. As described in §2.3, in the grading induced by $x_1 \in T_{p,q,r}$, each graded component of V_i is a representation of $\mathfrak{g}^{(x_1)} \times \mathbb{C}h_{x_1}$, in particular of $\mathfrak{g}^{(x_1)}$. The diagram $T - \{x_1\}$ consists of

$$y_{q-1} \text{ --- } \cdots \text{ --- } y_1 \text{ --- } u \text{ --- } z_1 \text{ --- } \cdots \text{ --- } z_{r-1},$$

which we take to be $\mathfrak{sl}(F_2)$, and

$$x_2 \text{ --- } \cdots \text{ --- } x_{p-1}$$

which we take to be $\mathfrak{sl}(F_0)$, so we may identify $\mathfrak{g}^{(x_1)}$ with $\mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2)$. The bottom graded components of V_3, V_2, V_1 are then F_2, F_2^*, F_0 respectively.

Theorem 4.5. *There exists a $\mathbb{C}$-algebra homomorphism $w_{\text{ssc}} \colon \widehat{R}_{\text{gen}} \to \mathbb{C}$ whose restrictions to the critical representations are the maps*

$$\begin{aligned} W(d_3) &= F_2^* \otimes [F_2 \oplus \cdots] \to \mathbb{C} \\ W(d_2) &= F_2 \otimes [F_2^* \oplus \cdots] \to \mathbb{C} \\ W(d_1) &= F_0^* \otimes [F_0 \oplus \cdots] \to \mathbb{C} \end{aligned}$$

given by the evident pairing in the bottom x_1-graded component, and zero on all higher graded components, and $\mathbb{F}_{\text{gen}} \otimes w_{\text{ssc}}$ is a split exact complex.

One striking feature of w_{ssc} is the use of the node $x_1 \in T$, and the lack of any mention of $z_1 \in T$. Recall that the latter node was involved in the defect Lie algebra $\mathbb{L}$ which was essential to the original construction of $\widehat{R}_{\text{gen}}$. But as we will see in §6, the node $x_1 \in T$ actually plays a more distinguished role in describing $\widehat{R}_{\text{gen}}$ retrospectively.

Proof. Let $C = \mathbb{C}^{r_2}$. Then the subgraph of T given by

$$y_{q-1} \text{ --- } \cdots \text{ --- } y_1 \text{ --- } u.$$

yields an inclusion $\mathfrak{sl}(C) \hookrightarrow \mathfrak{g}$. In particular we get decompositions $F_1 = F_0 \oplus C$ and $F_2 = F_1 \oplus C$, from which we assemble the following split exact complex:

$$\mathbb{F}_{\text{ssc}} \colon 0 \to F_3 \to F_3 \oplus C \to F_0 \oplus C \to F_0.$$

We view this as a graded complex where F_3 is in degree 1, C is in degree 0, and F_0 is in degree -1. From degree considerations analogous to Lemma 4.4, if we compute $w_1^{(3)}$ and $w_1^{(2)}$ respecting this grading, then all other higher structure maps are zero. We claim that this choice of higher structure maps gives the desired homomorphism w_{ssc} (note that by construction, $\mathbb{F}_{\text{gen}} \otimes w_{\text{ssc}} = \mathbb{F}_{\text{ssc}}$ is split exact). We have $M = \bigwedge^{f_1}(F_0 \oplus C) \otimes \bigwedge^{f_2}(F_3 \oplus C)^* \otimes \bigwedge^{f_3} F_3 = \bigwedge^{f_0} F_0$. The map

$$\beta \colon M^* \otimes \bigwedge^{r_1} F_1 = \bigwedge^{f_0} F_0^* \otimes \bigoplus_{k=0}^{f_0} \left(\bigwedge^{f_0-k} F_0 \otimes \bigwedge^{k} C \right) \to R$$

contracts $\bigwedge^{f_0} F_0^* \otimes \bigwedge^{f_0} F_0 \to R$ and is zero on all other factors.

Let $s_2 \colon F_1 \to F_2$ and $s_3 \colon F_2 \to F_3$ be the evident splittings of d_2 and d_3; that is, $s(x_0, c) = (0, c)$ for $x_0 \in F_0, c \in C$, and $s_3(x_3, c) = x_3$ for $x_3 \in F_3, c \in C$. These maps are homogeneous of degree zero, thus rather than "homogeneously lifting through d_i," we may just postcompose with s_i.

Hence we may take $w_1^{(3)}$ to just be the composite

$$M^* \otimes \bigwedge^{r_1+1} F_1 \to M^* \otimes \bigwedge^{r_1} F_1 \otimes F_1 \xrightarrow{\beta \otimes 1_{F_1}} F_1 \xrightarrow{s_2} F_2 = F_3 \oplus C.$$

This map sends the factor $\bigwedge^{f_0} F_0^* \otimes \bigwedge^{f_0} F_0 \otimes C$ identically to $C \subset F_2$, and is zero on all other factors. This can be verified directly, or one can just note that the map is $GL(F_0) \times GL(F_3) \times GL(C)$-equivariant and nonzero, and C is the only representation appearing in both source and target.

The bottom x_1-graded component of V_3 is $F_2 = F_3 \oplus C$. The former F_3 is the bottom z_1-graded component, mapping to F_2 via the differential $w_0^{(3)} = d_3$. The latter C comes from the next z_1-graded component, and we just saw how $w_1^{(3)}$ maps it to F_2. All other parts of the representation map to zero, so this proves that $w^{(3)}$ has the desired form.

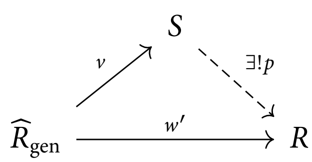
The proofs for $w^{(2)}$ and $w^{(1)}$ are completely analogous, so we omit them. □

5. Generic structure maps and the top complex

In existing literature such as [12] and [8], some of the structure maps $w_j^{(i)}$ have been computed by explicit lifts. When these lifts are not unique, additional variables are adjoined to parametrize the non-uniqueness to get "generic structure maps" $v_j^{(i)}$ which specialize to any particular choice of $w_j^{(i)}$ for the given resolution. We give a reformulation of Theorem 3.1 from this perspective.

Proposition 5.1. *Let $\mathbb{F}$ be a resolution over R. Let $(\widehat{R}_{\text{gen}}, \mathbb{F}^{\text{gen}})$ be the generic pair for the associated format, and $w\colon \widehat{R}_{\text{gen}} \to R$ a map which specializes $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}$. Define S to be the polynomial ring $R \otimes$ $\operatorname{Sym}(\bigoplus_{i>0} \mathbb{L}_i^*)$. We think of the adjoined variables as giving coordinates on the defect Lie algebra $\mathbb{L}$.*

Then there exists a map $v\colon \widehat{R}_{\text{gen}} \to S$ such that maps $w'\colon \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}$ correspond to R-algebra maps p making the below diagram commute.



Proof. For each i, $\mathbb{L}_i$ is finite-dimensional and there exists a "trace" element $u_i \in \mathbb{L}_i \otimes \mathbb{L}_i^*$. Explicitly if one takes a basis $e_1, \dots, e_n$ of $\mathbb{L}_i$ and $\epsilon_1, \dots, \epsilon_n$ its dual, then $u_i = \sum_{j=1}^n e_j \otimes \epsilon_j$.

Since $\mathbb{L} = \prod_{i>0} \mathbb{L}_i$, the infinite sum $X = \sum_{i>0} u_i$ is a well-defined element of $\mathbb{L} \otimes (\bigoplus_{i>0} \mathbb{L}_i^*) \subset \mathbb{L} \otimes S$, which we can think of as a "generic element" of $\mathbb{L}$. We define v to be the composite $w \exp X$ in the sense of Theorem 3.1.

The correspondence claimed in the proposition follows easily: the map p is just the data of an element of $\operatorname{Hom}_R(R \otimes \bigoplus_{i>0} \mathbb{L}_i^*, R) = \mathbb{L} \otimes R$ and the composite pv is the same as $w \exp p$ from this perspective, which reduces the statement to that of Theorem 3.1. □

The variables adjoined to R to obtain S are called *defect variables*, as they record the failure of higher structure maps being uniquely determined. Notice that the constructed v specializes $\mathbb{F}^{\text{gen}}$ to $\mathbb{F} \otimes S$, which is to say that $v_0^{(i)} = w_0^{(i)} \otimes S$.

Example 5.2. The ring S from Proposition 5.1 generalizes the ring defined in e.g. [12, Definition 3.1] for the formats of type D_n. There, the variables b_{ij}^k and c_{ut} adjoined are bases of $\mathbb{L}_1^* = \bigwedge^2 F_1 \otimes F_3^*$ and $\mathbb{L}_2^* = \bigwedge^4 F_1 \otimes \bigwedge^2 F_3^*$ respectively, which are the only nonzero components of $\mathbb{L}$ in that setting.

The explicit formulas for $v_j^{(i)}$ given subsequently in that paper can be recovered from Theorem 3.1 in the manner illustrated in Example 3.3, just taking X to instead be the "generic element" of $\mathbb{L} \otimes S$ as defined in Proposition 5.1 as opposed to any particular element of $\mathbb{L} \otimes R$.

The preceding discussion applies to all length three formats, but now we will consider a feature unique to the Dynkin case. For the sake of simplicity, suppose that $\mathfrak{g}$ for the Dynkin format under consideration has self-dual representations, i.e. the type is D_n (n even), E_7, or E_8. (We refer the reader to [18, Proposition 3.7] for adjustments in other cases.) In particular, their bottom graded components of the representations are dual to the top ones. So written as representations of $\prod SL(F_i)$, the decompositions of the critical representations are

$$W(d_3) = F_2^* \otimes V_-(-\omega_{z_{r-1}}) = F_2^* \otimes [F_3 \oplus \bigwedge^{r_0+1} F_1 \oplus \cdots \oplus F_3^*]$$
$$W(d_2) = F_2 \otimes V_-(-\omega_{y_{q-1}}) = F_2 \otimes [F_1^* \oplus F_3^* \otimes \bigwedge^{r_0} F_1 \oplus \cdots \oplus F_1]$$
$$W(d_1) = F_0^* \otimes V_-(-\omega_{x_{p-1}}) = F_0^* \otimes [F_1 \oplus F_3^* \otimes \bigwedge^{r_0+2} F_1 \oplus \cdots \oplus F_1^*]$$

Given $w\colon \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to some resolution $\mathbb{F}$, the bottom graded components $w_0^{(i)}$ of the critical representations give the differentials of $\mathbb{F}$. On the other hand, the restrictions of w to the top graded components give maps $w_{\text{top}}^{(3)}\colon F_3^* \to F_2$, $w_{\text{top}}^{(2)}\colon F_2 \to F_1^*$, and $w_{\text{top}}^{(1)}\colon F_1^* \to F_0$. Weyman observed that the symmetry of relations in $\widehat{R}_{\text{gen}}$ implies that these give the differentials of another complex:

$$\mathbb{F}^{\text{top}}\colon 0 \to F_3^* \to F_2 \to F_1^* \to F_0.$$

As the higher structure maps $w_{>0}^{(i)}$ are not uniquely determined by $\mathbb{F}$, neither is the complex $\mathbb{F}^{\text{top}}$. It is conjectured that if $\mathbb{F}$ resolves a Cohen-Macaulay R-module, then there exists a choice of $\mathbb{F}^{\text{top}}$ that is split exact. The significance of this claim is that, by symmetry of $\widehat{R}_{\text{gen}}$, $\mathbb{F}$ could be viewed as a particular choice of "top complex" for a split exact complex. In other words, it would be a specialization of the generic such top complex defined over the polynomial ring $S = \operatorname{Sym}(\bigoplus_{i>0} \mathbb{L}_i^*)$ from Proposition 5.1.

Theorems 4.1 and 5.1 in [12] describe the maps $v_{\text{top}}^{(i)}$ for a split exact complex in the case of formats $(1, n, n, 1)$ and $(1, 4, n, n-3)$. In that paper they are computed by lifting. With our results we can now provide an alternative construction.

Theorem 5.3. *For the split exact complex $\mathbb{F}_{\text{ssc}}$ from Theorem 4.5, the generic structure maps $v_{\text{top}}^{(i)}$ can be computed as follows. Let $X \in \mathbb{L} \otimes S$ be the generic element of $\mathbb{L}$. Then $v_{\text{top}}^{(3)}$ is the composite*

$$S \otimes F_3^* \xrightarrow{i_{z_1}^{\text{top}}} S \otimes V_-(-\omega_{z_{r-1}}) \xrightarrow{\exp X} S \otimes V_-(-\omega_{z_{r-1}}) \xrightarrow{p_{x_1}^{\text{bottom}}} S \otimes F_2,$$

$v_{\text{top}}^{(2)}$ is the composite

$$S \otimes F_1 \xrightarrow{i_{z_1}^{\text{top}}} S \otimes V_-(-\omega_{y_{q-1}}) \xrightarrow{\exp X} S \otimes V_-(-\omega_{y_{q-1}}) \xrightarrow{p_{x_1}^{\text{bottom}}} S \otimes F_2^*,$$

and $v_{\text{top}}^{(1)}$ is the composite

$$S \otimes F_1^* \xrightarrow{i_{z_1}^{\text{top}}} S \otimes V_-(-\omega_{x_{p-1}}) \xrightarrow{\exp X} S \otimes V_-(-\omega_{x_{p-1}}) \xrightarrow{p_{x_1}^{\text{bottom}}} S \otimes F_0.$$

Here $i_{z_1}^{\text{top}}$ means the inclusion of the top z_1-graded piece and $p_{x_1}^{\text{bottom}}$ means projection onto the bottom x_1-graded piece.

Proof. In each composite, the third map is w_{ssc} as described in Theorem 4.5. Precomposing with $\exp X$ gives the generic choice of structure maps by Proposition 5.1, and then we restrict to the top z_1-graded component to get $v_{\text{top}}^{(i)}$. □

This construction is studied explicitly in [13] for the various Dynkin cases. There it is illustrated how, for type D_n, it recovers the complexes described in [12]. The explanation for this apparent coincidence was not given in that paper, but now it is explained by Theorem 5.3.

The complex constructed for $(1, n, n, 1)$ visibly agrees with the generic Buchsbaum-Eisenbud example [5]. For $(1, 4, n, n-3)$, §3.1.3 in [13] recovers the generic example for an almost complete interesction (see [5], [1]) and explains how it is in some sense equivalent to the complex in [12, Theorem 5.1]. With Theorem 5.3, we can explain this observed equivalence in another way: [12, Theorem 5.1] describes the generic top complex for a split complex of format $(1, 4, n, n-3)$, while the others give the generic top complex for the dual format $(n-3, n, 4, 1)$. Since these are generic examples for resolutions of the given format with acyclic dual, it stands to reason that they must be equivalent!

For the format $(1, 5, 6, 2)$ of type E_6, the generic top complex for a split exact complex is constructed in [6]. To be precise, the top complex for $(2, 6, 5, 1)$ is constructed there, but again this is equivalent to the one for $(1, 5, 6, 2)$. This equivalence is briefly described at the end of [6, §6] and more explicitly in [13, Theorem 2.5]. The complex is reproduced in [13] via the construction of Theorem 5.3: compare the matrices given in [6, §3] to those in [13, §3.2].

In [13], it is also discussed how the constructed complexes resolve coordinate rings of certain Schubert varieties restricted to an affine patch, relating the construction to resolutions studied in [15]. This connection with Schubert varieties will be further explained in §6.1.

6. Defining relations of $\widehat{R}_{\text{gen}}$

We will now make use of the results from §3 and §4 to analyze the generators and relations of $\widehat{R}_{\text{gen}}$. In other words, we will describe it as a quotient of a polynomial ring over $\mathbb{C}$. For the generators we will need to consider two more representations of $\mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{g}$ in addition to the representations $W(d_i)$, namely those generated by the entries of the Buchsbaum-Eisenbud multipliers a_2 and a_1. We will call these $W(a_2)$ and $W(a_1)$ respectively. Altogether we have

$$\begin{aligned}
W(d_3) &= F_2^* \otimes V_-(-\omega_{z_{r-1}}) \\
&= F_2^* \otimes [F_3 \oplus \cdots] \\
W(d_2) &= F_2 \otimes V_-(-\omega_{y_{q-1}}) \\
&= F_2 \otimes [F_1^* \oplus \cdots] \\
W(d_1) &= F_0^* \otimes V_-(-\omega_{x_{p-1}}) \\
&= F_0^* \otimes [F_1 \oplus \cdots] \\
W(a_2) &= \bigwedge^{f_2} F_2 \otimes V_-(-\omega_{x_1}) \\
&= \bigwedge^{f_2} F_2 \otimes [\bigwedge^{r_2} F_1^* \otimes \bigwedge^{f_3} F_3^* \oplus \cdots] \\
W(a_1) &= \bigwedge^{f_0} F_0^* \otimes \bigwedge^{f_1} F_1 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_3} F_3
\end{aligned}$$

As in §3, $V_-(-\lambda)$ denotes the irreducible $\mathfrak{g}$-representation with lowest weight $-\lambda$. When $T_{p,q,r}$ is a Dynkin diagram these are finite-dimensional and isomorphic to $V(\lambda)^*$. Here we have displayed the bottom z_1-graded components, which are the entries of d_i and a_i. Note that the representation $W(a_1)$ is a trivial representation of $\mathfrak{g}$, though not of $\prod \mathfrak{gl}(F_i)$. Let A_1 denote the direct sum of the above five representations. The next result was communicated to the author by Jerzy Weyman.

Proposition 6.1. *A_1 generates $\widehat{R}_{\text{gen}}$ as a $\mathbb{C}$-algebra.*

Proof. The entries of d_i, a_i generate the Buchsbaum-Eisenbud multiplier ring R_a. From the explicit decomposition of $\widehat{R}_{\text{gen}}$ given in [17], one sees that every irreducible representation of $\mathfrak{sl}(F_0)\times\mathfrak{sl}(F_2)\times \mathfrak{g}$ in $\widehat{R}_{\text{gen}}$ has bottom z_1-graded component belonging to the subring R_a; this is just the statement that R_a is the subring of $\widehat{R}_{\text{gen}}$ on which $\mathbb{L}$ acts trivially.

In particular, this means that R_a generates $\widehat{R}_{\text{gen}}$ as a representation of $\mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{g}$. This Lie algebra acts on $\widehat{R}_{\text{gen}}$ by derivations and A_1 is closed under this action. It follows that the subring generated by A_1 is also closed under this action. Since it contains R_a, it must be the entire ring $\widehat{R}_{\text{gen}}$ as claimed. □

Next we analyze the relations which hold among the elements of A_1. More precisely, let $A = \operatorname{Sym}_{\mathbb{C}} A_1$, let $\pi\colon A \to \widehat{R}_{\text{gen}}$ be the evident quotient map, and let $\hat{I} = \ker \pi$ be the ideal of defining relations for $\widehat{R}_{\text{gen}}$.

The main tool for analyzing the ideal $\hat{I}$ is the following result from [16], which we slightly paraphrase as follows:

Lemma 6.2 ([16, Lemma 2.4])**.** *The relations which hold in $\widehat{R}_{\text{gen}}$ are exactly those which hold for arbitrary choices of higher structure maps for split exact complexes.*

For example, this result was used heavily in [8] to directly verify explicit relations on higher structure maps, many of which were checked with computer assistance.

Here we will make use of this result in a more conceptual manner. Theorem 4.5 provides an explicit description of a particular choice of higher structure maps for a split exact complex. Moreover, Theorem 3.1 shows how the exponential action of the defect Lie algebra can be used to parametrize all choice of higher structure maps for a given resolution. Hence we may reformulate the previous result in the following way.

Lemma 6.3. *The ideal $\hat{I} \subset A$ is the largest ideal in $\ker(w_{\text{ssc}}\pi)$ that is closed under the actions of $\mathbb{L}$ and $\mathfrak{gl}(F_i)$ for all i.*

The ring $\widehat{R}_{\text{gen}}$ is a domain for all formats, and if $T_{p,q,r}$ is a Dynkin diagram, $\widehat{R}_{\text{gen}}$ is a finite-type $\mathbb{C}$-algebra. Then this can be interpreted as saying that $\operatorname{Spec}\widehat{R}_{\text{gen}}$ is the orbit closure of the $\mathbb{C}$-point corresponding to $w_{\text{ssc}}\pi$ (in the affine space $\operatorname{Spec} A$) under the actions of $\exp\mathbb{L}$ and $GL(F_i)$.

The main technical goal of this section is to unify the actions of $\prod \mathfrak{gl}(F_i)$ and $\mathbb{L}$, so that we can instead consider the action of a single Lie algebra $\hat{\mathfrak{g}}$. The Lie algebra $\mathfrak{sl}(F_2) \times \mathfrak{sl}(F_0) \times \mathfrak{g}$ does not quite suffice, since although the algebras $\mathfrak{sl}(F_i)$ are present here, nonzero scalars in $\mathfrak{gl}(F_i)$ are not.

We define an action of the abelian Lie algebra $\mathfrak{t} = \mathbb{C}^3$ on A_1 as follows. Note that the action of a Lie algebra element on A_1 uniquely extends to a derivation on the entirety of A via the Leibniz rule, so we will restrict our attention to A_1. An element $(\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{t}$ acts as multiplication by λ_i on $W(d_i)$, $r_2\lambda_2 - r_3\lambda_3$ on $W(a_2)$, and $r_1\lambda_1 - r_2\lambda_2 + r_3\lambda_3$ on $W(a_1)$.

Recall from §2.3 that the middle z_1-graded component $\ker(\operatorname{ad} h_{z_1}) \subset \mathfrak{g}$ has the form

$$\mathfrak{sl}(F_3) \times \mathfrak{sl}(F_1) \times \mathbb{C}h_{z_1}$$

unless $T_{p,q,r}$ is one of the affine Dynkin diagrams $\widetilde{E}_n$. (If $T_{p,q,r} = \widetilde{E}_n$, then there is an extra copy of $\mathbb{C}$ in this middle component, but it will not affect the following setup.) Let

$$\hat{\mathfrak{g}} = \mathfrak{sl}(F_2) \times \mathfrak{sl}(F_0) \times \mathfrak{g} \times \mathfrak{t}.$$

We will view $\mathfrak{gl}(F_i) = \mathfrak{sl}(F_i) \times \mathbb{C}$ as subalgebras of $\hat{\mathfrak{g}}$ via inclusions ι_i, which we now define. Let τ_i denote the smallest eigenvalue of h_{z_1} acting on $W(d_i)$, and $\underline{\tau} = (\tau_1, \tau_2, \tau_3) \in \mathfrak{t}$. The map ι_i sends $\mathfrak{sl}(F_i)$ to itself, and $1 \in \mathbb{C}$ to an element of $\mathbb{C}h_{z_1} \times \mathfrak{t}$ as follows:

$$\begin{aligned}
\iota_0(1) &= (-1, 0, 0)\\
\iota_1(1) &= (r_2 - 1)h_{z_1} + (1, -1, 0) - (r_2 - 1)\underline{\tau}\\
\iota_2(1) &= f_2 h_{z_1} + (0, 1, -1) - f_2\underline{\tau}\\
\iota_3(1) &= -(1 + f_3)h_{z_1} + (0, 0, 1) + (1 + f_3)\underline{\tau}.
\end{aligned}$$

Theorem 6.4. *The Lie algebra $\hat{\mathfrak{g}}$ acts on A_1, extending the actions of $\mathfrak{gl}(F_i) \subset \hat{\mathfrak{g}}$. If $T_{p,q,r}$ is Dynkin, the subalgebras $\mathfrak{gl}(F_i)$ and $\mathfrak{g}$ generate the entirety of $\hat{\mathfrak{g}}$.*

We conjecture that the statement should hold even without the Dynkin hypothesis.

Proof. We already have $\mathfrak{sl}(F_2)\times\mathfrak{sl}(F_0)\times\mathfrak{g}$ acting on A_1, and this commutes with the action of $\mathfrak{t}$ since elements of $\mathfrak{t}$ act via scalars on each of the five representations in A_1. So their product $\hat{\mathfrak{g}}$ does act on A_1.

This evidently extends the actions of $\mathfrak{sl}(F_i)$, so to establish the first claim it is sufficient to take $1 \in \mathbb{C} \subset \mathfrak{gl}(F_i)$ and verify that 1 and $\iota_i(1)$ act the same way on A_1. This can be checked on each z_1-graded component of each representation.

As an example, we check it for $1 \in \mathfrak{gl}(F_1)$ on $W(d_2)$. Since the bottom graded component of $W(d_2)$ is $F_2 \otimes F_1^*$ and the part of $\mathfrak{g}$ in degree 1 is dual to

$$\mathbb{L}_1 = \bigwedge^{r_1+1} F_1^* \otimes F_3 \otimes \bigwedge^{f_1} F_1 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_3} F_3,$$

we see that $1 \in \mathfrak{gl}(F_1)$ acts by $-1 + (f_1 - r_1 - 1)k = -1 + (r_2 - 1)k$ on the k-th graded component of $W(d_2)$, where we index so that $k = 0$ refers to the bottom component. The element h_{z_1} acts by $\tau_2 + k$ on the k-th graded component, thus the action of $i_1(1) \in \hat{\mathfrak{g}}$ is

$$(r_2 - 1)(\tau_2 + k) + (-1) - (r_2 - 1)\tau_2 = -1 + (r_2 - 1)k$$

which agrees. The verification for any other $\mathfrak{gl}(F_i)$ on $W(d_j)$ is similar, so we omit it.

The verification for $W(a_2)$ and $W(a_1)$ is almost the same, except one needs to know how the eigenvalues of h_{z_1} on those representations relate to τ_1, τ_2, τ_3. This is given in Lemma 6.5, which one should compare to the action of $\mathfrak{t}$ on $W(a_2)$ and $W(a_1)$ defined before.

For the final statement of the theorem, it suffices to check that the vectors

$$(-1, 0, 0), \quad (1, -1, 0) - (r_2 - 1)\underline{\tau}, \quad (0, 1, -1) - f_2\underline{\tau}, \quad (0, 0, 1) + (1 + f_3)\underline{\tau}$$

span $\mathfrak{t}$. This has been explicitly verified for each Dynkin format, but we lack a systematic way of proving it in general. □

Lemma 6.5. *The lowest eigenvalue of h_{z_1} acting on $W(a_2)$ is $r_1\tau_1$, which is equal to $r_2\tau_2 - r_3\tau_3$. In particular $0 = r_1\tau_1 - r_2\tau_2 + r_3\tau_3$ is the lowest (and only) eigenvalue of h_{z_1} acting on $W(a_1)$.*

Proof. The lowest eigenvalue of h_{z_1} on $V_-(-\omega_{x_{p-1}})$ is τ_1 by definition. The corresponding eigenspace (the bottom z_1-graded component) has dimension $f_1 \geq r_1$, so the lowest eigenvalue for $\bigwedge^{r_1} V_-(-\omega_{x_{p-1}})$ is $r_1\tau_1$. The lowest weight appearing in $\bigwedge^{r_1} V_-(-\omega_{x_{p-1}})$ is

$$-\omega_{x_{p-1}} + (\omega_{x_{p-1}} - \omega_{x_{p-2}}) + \cdots + (\omega_{x_2} - \omega_{x_1}) = -\omega_{x_1}$$

recalling that $p = r_1 + 1$. This shows that the lowest eigenvalue of h_{z_1} on $V_-(-\omega_{x_1})$, and thus on $W(a_2)$, is $r_1\tau_1$.

Similarly one can show that the lowest eigenvalue of h_{z_1} on $V_-(-\omega_{z_1})$ is $r_3\tau_3$. So the lowest eigenvalue on $V_-(-\omega_{x_1}) \otimes V_-(-\omega_{z_1})$ is $r_1\tau_1 + r_3\tau_3$. The lowest weight appearing in this tensor product is $-\omega_{x_1} - \omega_{z_1}$, but this is also the lowest weight appearing in $\bigwedge^{r_2} V_-(-\omega_{y_{q-1}})$:

$$-\omega_{y_{q-1}} + (\omega_{y_{q-1}} - \omega_{y_{q-2}}) + \cdots + (\omega_{y_1} - \omega_u) + (\omega_u - \omega_{x_1} - \omega_{z_1}) = -\omega_{x_1} - \omega_{z_1}$$

recalling that $q = r_2 - 1$. So the lowest eigenvalue of h_{z_1} on $V_-(-\omega_{x_1}) \otimes V_-(-\omega_{z_1})$ is the same as on $\bigwedge^{r_2} V_-(-\omega_{y_{q-1}})$, proving $r_1\tau_1 + r_3\tau_3 = r_2\tau_2$. The final statement in the lemma is just the observation that $W(a_1)$ is a trivial representation of $\mathfrak{g}$. □

Example 6.6. Revisiting $T_{2,3,3}$ for the format $(1,5,6,2)$ as in Example 2.1,

- $V_-(-\omega_{x_1})$ is the adjoint and the action of h_{z_1} has eigenvalues $-2,-1,0,1,2$,
- $V_-(-\omega_{y_2}) = V(\omega_{z_2})$ has eigenvalues $-4/3, -1/3, 2/3, 5/3$, and
- $V_-(-\omega_{z_2})$ has eigenvalues $-5/3, -2/3, 1/3, 4/3$.

We indeed have the identity

$$r_1\tau_1 - r_2\tau_2 + r_3\tau_3 = (1)(-2) - (4)(-4/3) + (2)(-5/3) = 0.$$

Using Theorem 6.4 we can revisit Lemma 6.3 and give some alternative characterizations of the ideal of relations $\hat{I} = \ker \pi$.

Theorem 6.7. *If $T_{p,q,r}$ is Dynkin, the following are equivalent characterizations of the ideal $\hat{I} \subset A$ of defining relations for $\widehat{R}_{\mathrm{gen}}$.*

1. *$\hat{I}$ is the largest ideal in $\ker(w_{\mathrm{ssc}}\pi)$ that is closed under the actions of $\mathbb{L}$ and $\mathfrak{gl}(F_i)$ for $i = 0,1,2,3$.*
2. *$\hat{I}$ is the largest ideal in $\ker(w_{\mathrm{ssc}}\pi)$ that is closed under the actions of $\mathfrak{g}$ and $\mathfrak{gl}(F_i)$ for $i = 0,1,2,3$.*
3. *$\hat{I}$ is the largest ideal in $\ker(w_{\mathrm{ssc}}\pi)$ that is closed under the action of $\hat{\mathfrak{g}} = \mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{g} \times \mathfrak{t}$.*
4. *$\hat{I}$ is the largest ideal in $\ker(w_{\mathrm{ssc}}\pi)$ that is closed under the action of $\mathfrak{g} \times \mathfrak{t}$.*

Proof. Characterization (1) is Lemma 6.3. Any ideal closed under $\mathfrak{g}$ is certainly closed under $\bigoplus_i \mathbb{L}_i \subset \mathfrak{g}$. As we are assuming $T_{p,q,r}$ is Dynkin, we have $\mathbb{L} = \prod_i \mathbb{L}_i = \bigoplus_i \mathbb{L}_i$, but even without this assumption, closure under $\bigoplus_i \mathbb{L}_i$ and $\prod_i \mathbb{L}_i$ are equivalent since every element of A is killed by $\mathbb{L}_i$ for $i \gg 0$. As π is $\mathfrak{g}$-equivariant, the ideal $\hat{I}$ is closed under $\mathfrak{g}$, and thus (2) holds. The equivalence of (2) and (3) follows from Theorem 6.4.

To show that (3) and (4) are equivalent, it will be more convenient to pass to a group action instead. Let $p_{\mathrm{ssc}} \in \operatorname{Spec} A$ be the point corresponding to $w_{\mathrm{ssc}}\pi$. Let G be the group associated to the Lie algebra $\mathfrak{g}$, so $G \times \mathbb{C}^3$ corresponds to $\mathfrak{g} \times \mathfrak{t}$.

There is a group homomorphism $\eta\colon SL(F_2) \times SL(F_0) \to G$ induced by the inclusion of the diagram $T - \{x_1\}$ inside of T. In general, if $g \in SL(F_2) \times SL(F_0)$, the actions of g and $\eta(g)$ on $\operatorname{Spec} A$ are very different, since they act on different tensor factors in A_1. However, the actions of $SL(F_2) \times SL(F_0)$ and $G \times \mathbb{C}^3$ commute, and for the point p_{ssc} in particular, we have $g \cdot p_{\mathrm{ssc}} = \eta(g) \cdot p_{\mathrm{ssc}}$ from the explicit description in Theorem 4.5. Hence if $g' \in G \times \mathbb{C}^3$, the action of $(g, g') \in SL(F_2) \times SL(F_0) \times G \times \mathbb{C}^3$ on p_{ssc} is the same as the action of $g'\eta(g) \in G \times \mathbb{C}^3$. It follows that the following two orbits coincide:

$$(SL(F_2) \times SL(F_0) \times G \times \mathbb{C}^3)p_{\mathrm{ssc}} = (G \times \mathbb{C}^3)p_{\mathrm{ssc}}.$$

Taking the ideal of functions which vanish on these orbits, we get the equivalence of (3) and (4). □

Either of characterizations (3) or (4) will suffice for the applications in the remainder of this section. Note that the theorem also holds if one instead takes A to be the polynomial ring generated by some subrepresentation of A_1. By doing so, we may investigate relations involving only a subset of the five representations.

6.1. $\widehat{R}_{\text{gen}}$ **and Schubert varieties.** One very important example is obtained by taking the representation $W(a_2)$ alone. Let $R(a_2)$ denote the subring of $\widehat{R}_{\text{gen}}$ generated by this representation.

Theorem 6.8. *Assume that $T_{p,q,r}$ is Dynkin. Let G be the group associated to the Lie algebra $\mathfrak{g}$. The subring $R(a_2) \subset \widehat{R}_{\text{gen}}$ is the homogeneous coordinate ring of G/P_{x_1} in the Plücker embedding $G/P_{x_1} \hookrightarrow \mathbb{P}(V(\omega_{x_1}))$, where $P_{x_1} \subset G$ is the maximal parabolic subgroup for $x_1 \in T_{p,q,r}$.*

Let $v \in V(\omega_{x_1})$ be a highest weight vector, and let $B \subset G$ be the subgroup corresponding to the Borel $\mathfrak{b}_-$. Write X^w for the codimension three Schubert variety $\overline{Bw \cdot [v]} \subset \mathbb{P}(V(\omega_{x_1}))$, where $w = s_{z_1} s_u s_{x_1}$. Then the entries of a_2 cut out $X^w \subset G/P_{x_1}$ set-theoretically.

Proof. Let $A = \operatorname{Sym} W(a_2)$. As a representation of $\mathfrak{g}$, $W(a_2) = V_-(-\omega_{x_1}) = V(\omega_{x_1})^*$. So we may view $\operatorname{Spec} A$ as the affine space $V(\omega_{x_1})$. Let $\pi \colon A \to R(a_1) \subset \widehat{R}_{\text{gen}}$ be the evident map. To understand the point of $\operatorname{Spec} A = V(\omega_{x_1})$ described by $w_{\text{ssc}}\pi$, we revisit Theorem 4.5. Although the restriction of w_{ssc} to $W(a_2)$ was not explicitly written there, it is easily inferred: $W(a_2) \otimes W(a_1)$ is the extremal subrepresentation inside $\bigwedge^{f_0} F_0^* \otimes \bigwedge^{f_0} V_-(-\omega_{x_{p-1}}) \subset S_{f_0} W(d_1)$, and since $W(a_1)$ is just the scalar a_1, we deduce that the restriction of w_{ssc} to $W(a_2)$ is nonzero only on the lowest weight space of the representation.

Geometrically, this means $\ker(w_{\text{ssc}}\pi)$ defines a highest weight vector $v \in V(\omega_{x_1})$. Using either (3) or (4) of Theorem 6.7 (as the action of $SL(F_2) \times SL(F_0)$ is trivial here), we conclude that $\operatorname{Spec} R(a_1) \subset V(\omega_{x_1})$ is the orbit closure of v under the action of $G \times \mathbb{C}^3$ where $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ acts by $\exp(-r_1\lambda_1 + r_2\lambda_2 - r_3\lambda_3)$. Of course, it is equivalent to consider the orbit closure under $G \times \mathbb{C}^\times$ where $\mathbb{C}^\times$ acts by scaling.

Thus $\operatorname{Spec} R(a_1)$ is the affine cone over $G \cdot [v] \subset \mathbb{P}(V(\omega_{x_1}))$. The stabilizer of the highest weight line $[v] \in \mathbb{P}(V(\omega_{x_1}))$ is the parabolic $P_{x_1} \subset G$, so the first part of the theorem follows.

Let $\varphi \in W(a_2)$ be a lowest weight vector, i.e. the Plücker coordinate dual to $[v]$. Let W be the Weyl group of G, $W_{P_{x_1}}$ the subgroup generated by all reflections other than s_{x_1}, and $W^{P_{x_1}}$ the set of minimal length representatives of $W/W_{P_{x_1}}$. The extremal Plücker coordinates which set-theoretically cut out X^w for $w = s_{z_1} s_u s_{x_1}$ are given by $\sigma \cdot \varphi$ for $\sigma \in W^{P_{x_1}}$ such that $\sigma \ngeq w$ in the partial Bruhat order. Such representatives are exactly those which do not involve the reflection s_{z_1}. There are $\binom{f_1}{r_2}$ of these, corresponding to the extremal weights of $\bigwedge^{r_2} F_1^*$, the bottom graded component of $W(a_2)$.

By definition, X^w is the B-orbit closure of $w \cdot [v]$, where $B \subset G$ corresponds to the Borel subalgebra $\mathfrak{b}_-$. The linear span of X^w is the representation of $\mathfrak{b}_-$ generated by $w \cdot [v]$. It is easy to check that the vector $w \cdot v \in V(\omega_{x_1})$ is killed by all $e_i \in \mathfrak{g}$ other than e_{z_1}, thus we may replace $\mathfrak{b}_-$ by the (negative) maximal parabolic subalgebra $\mathfrak{p}_{z_1}$. In particular, the linear span of (the cone over) X^w is a representation of $\mathfrak{g}^{(z_1)} = \mathfrak{sl}(F_3) \times \mathfrak{sl}(F_1)$, and it does not meet the top z_1-graded component of $V(\omega_{x_1})$. So all elements of $\bigwedge^{r_2} F_1^* \subset W(a_2)$, i.e. entries of a_2, must vanish on X^w. ☐

Remark 6.9. It is well-known from the theory of Demazure modules that Schubert varieties are cut out ideal-theoretically by Plücker coordinates, not just set-theoretically. But in order to conclude that the entries of a_2 cut out X^w ideal-theoretically, it is necessary to know that the second z_1-graded component of $W(a_2)$ is irreducible, as this is equivalent to the linear span of X^w containing all components other than the top one. The author has not verified this for all Dynkin formats at the time of writing, although it is true for formats where $f_0 = 1$.

This theorem is another step in understanding the deep connection between Weyman's generic free resolutions and Schubert varieties. Previously, in [15], these Schubert varieties were studied for formats $(1, f_1, f_2, f_3)$ of type E_n. The free resolutions of their coordinate rings were inferred using linkage. In [13] these examples were revisited and expanded upon, and it was shown how the differentials in the free resolutions given in [15] could be explicitly described using representations of $\mathfrak{g}$. The interest in these examples stems from a conjecture [18, 4.9] that they give generic resolutions of perfect ideals, c.f. the discussion at the end of §5.

Conjecture 1.2, a refinement of the preceding, will be stated and proven in a forthcoming paper. We will show that if $\mathbb{F}$ resolves R/I for a perfect ideal I, then any map $\widehat{R}_{\text{gen}} \to R$ specializing the generic resolution to $\mathbb{F}$ results in a map whose restriction to $W(a_2)$ is nonzero modulo the maximal ideal $\mathfrak{m} \subset R$. From the perspective of Theorem 6.8, if we write $k = R/\mathfrak{m}$ for the residue field, we get a map $\operatorname{Spec} k \to \operatorname{Spec} R(a_1)$ which lands in the complement of the cone vertex. The structure of I is related to where this k-valued point resides in the Schubert cell stratification of G/P_{x_1}.

6.2. $\widehat{R}_{\text{gen}}$ **for linked formats.** Suppose a complex $\mathbb{F}$ of format $(1, f_1, f_2, f_3)$ resolves R/I for a perfect ideal I in a local Gorenstein $\mathbb{C}$-algebra R. If $s_1, s_2, s_3 \in I$ is a regular sequence among a minimal set of generators for I, then $R/((s_1, s_2, s_3) : I)$ can be resolved by a complex of format $(1, f_3 + 3, f_2, f_1 - 3)$ [14, Prop. 2.6]. For this reason we say these two formats are *linked*. More generally, given a format $\underline{f} = (f_0, f_1, f_2, f_3)$, we say the linked[3] format is $\underline{f}' = (f_0, f_3 + f_0 + 2, f_2, f_1 - f_0 - 2)$.

We will use Theorem 4.5 and Theorem 6.7 to compare the generic rings for the two formats. We will write $\widehat{R}_{\text{gen}}$ for the generic ring associated to $\underline{f}$ and $\widehat{R}'_{\text{gen}}$ for the one associated to $\underline{f}'$. In what follows, if $(-)$ denotes some construction for $\underline{f}$, $(-)'$ denotes the same construction for $\underline{f}'$.

Theorem 6.10. *Fix a Dynkin format $\underline{f} = (f_0, f_1, f_2, f_3)$ and its linked format $\underline{f}'$. Let $R(d_{1,2,3})$ denote the subring of $\widehat{R}_{\text{gen}}$ generated by $\bigoplus_i W(d_i)$, and similarly for $R(d_{1,2,3})' \subset \widehat{R}'_{\text{gen}}$. Then the rings $R(d_{1,2,3})$ and $R(d_{1,2,3})'$ are isomorphic.*

Proof. For linked formats, the respective graphs $T_{p,q,r}$ are effectively the same, except the y and z arms are interchanged so $y'_k = z_k$ with the preceding notation. Indeed, if we let $p = f_0+1$, $q = f_1-f_0-1$, and $r = f_3 + 1$, then $q = (f_1 - f_0 - 2) + 1 = f'_3 + 1$ and $r = (f_3 + f_0 + 2) - f_0 - 1 = f'_1 - f'_0 - 1$, so $\mathfrak{g} = \mathfrak{g}'$ and there is no ambiguity when we write representations $V_-(-\lambda)$. Inside of $\widehat{R}_{\text{gen}}$ we have the critical representations

$$
\begin{aligned}
W(d_3) &= F_2^* \otimes V_-(-\omega_{z_{r-1}})\\
W(d_2) &= F_2 \otimes V_-(-\omega_{y_{q-1}})\\
W(d_1) &= F_0^* \otimes V_-(-\omega_{x_{p-1}}).
\end{aligned}
$$

Inside of $\widehat{R}'_{\text{gen}}$ we have

$$
\begin{aligned}
W(d_3)' &= F_2'^* \otimes V_-(-\omega_{z'_{q-1}}) = F_2'^* \otimes V_-(-\omega_{y_{q-1}})\\
W(d_2)' &= F'_2 \otimes V_-(-\omega_{y'_{r-1}}) = F'_2 \otimes V_-(-\omega_{z_{r-1}})\\
W(d_1)' &= F_0'^* \otimes V_-(-\omega_{x'_{p-1}}) = F_0'^* \otimes V_-(-\omega_{x_{p-1}}).
\end{aligned}
$$

Let A be the polynomial ring on $\bigoplus_i W(d_i)$, and $R(d_{1,2,3}) \subset \widehat{R}_{\text{gen}}$ the subring generated by these representations. Let $\pi\colon A \to R(d_{1,2,3}) \subset \widehat{R}_{\text{gen}}$ be the evident map. Define A', $R(d_{1,2,3})'$, and π' analogously.

We see that if we identify F'_2 with F_2^* and F'_0 with F_0, then we can identify $W(d_3)'$ with $W(d_2)$, $W(d_2)'$ with $W(d_3)$, and $W(d_1)'$ with $W(d_1)$. Moreover, the maps $w_{\text{ssc}}\pi$ and $w'_{\text{ssc}}\pi'$ from Theorem 4.5 coincide with this identification. This is the essence of the comment after Theorem 4.5:

[3] If $f_0 > 1$ then one should consider Buchsbaum-Rim linkage in place of ordinary linkage.

although $z_1 \neq y_1 = z_1'$ on $T_{p,q,r}$, the description of $w_{ssc}\pi$ does not involve that vertex. Rather, it revolves around the grading induced by $x_1 = x_1'$, which is "shared" between the two formats.

By combining this with either (3) or (4) of Theorem 6.7, we conclude that the defining ideals of $R(d_{1,2,3})$ and $R(d_{1,2,3})'$ are the same in $A = A'$, since the Lie algebra actions coincide as well. □

In a forthcoming paper, we intend to use Theorem 6.10 to prove Conjecture 1.1: if $(1, f_1, f_2, f_3)$ is a Dynkin format, then the ideal I—with hypotheses as laid out at the beginning of this subsection—is in the linkage class of a complete intersection ("licci").

The main idea of the proof is that the specialization $\widehat{R}_{\text{gen}} \to R$ restricts to a map $R(d_{1,2,3}) \to R$ that can be interpreted in *two* ways owing to the identification $R(d_{1,2,3}) \cong R(d_{1,2,3})'$ from Theorem 6.10. A careful examination shows that under mild hypotheses, this yields a pair of resolutions $\mathbb{F}, \mathbb{F}'$ which resolve linked ideals I, I'. Then the action of G on $R(d_{1,2,3})$ is used to repeatedly adjust this pair of ideals, keeping one fixed at each step. If the format of $\mathbb{F}$ is Dynkin, then this process terminates with I or I' generated by a regular sequence.

References

[1] Brown, A. *A structure theorem for a class of grade three perfect ideals,* J. Algebra 105 (1987), no. 2, 308–327

[2] Bruns, W. *The existence of generic free resolutions and related objects,* Math. Scand. 55 (1984) 33-46

[3] Buchsbaum, D., Eisenbud, D. *What makes a complex exact,* J. of Algebra. 25 (1973) 259-268

[4] Buchsbaum, D., Eisenbud, D. *Some structure theorems for finite free resolutions,* Advances in Math. 1 (1974) 84-139

[5] Buchsbaum, D., Eisenbud, D. *Algebra structures for finite free resolutions and some structure theorems for ideals of codimension 3,* American J. of Math. 99, no. 3 (1977) 447-485

[6] Celikbas, E., Laxmi, J., Kraśkiewicz, W., Weyman, J. *The family of perfect ideals of codimension 3, of type 2 with 5 generators,* Proc. Amer. Math. Soc. 148 (2020), 2745-2755

[7] Christensen, L. W., Veliche, O., Weyman, J. *Free resolutions of Dynkin format and the licci property of grade perfect ideals.* Mathematica Scandinavica, 125(2), 163–178. (2019)

[8] Guerrieri, L., Ni, X., Weyman, J. *Higher structure maps for free resolutions of length 3 and linkage,* `https://arxiv.org/abs/2208.05934`

[9] Hochster, M. *Topics in the homological theory of modules over commutative rings,* CBMS Regional Conference Series in Mathematics, vol. 24, American Mathematical Society, (1975)

[10] Huneke, C., Ulrich, B. *The structure of linkage,* Ann. of Math. (2) 126 (1987), no.2,277-334,

[11] Lee K., Weyman J. *Some branching formulas for Kac-Moody Lie algebras,* Commun. Korean Math. Soc.34 (2019), No. 4, pp. 1079-1098

[12] Guerrieri, L., Weyman, J. *Mapping free resolutions of length three I,* preprint, arXiv:2012.08358, (2020)

[13] Ni, X., Weyman, J. *Free resolutions constructed from bigradings on Lie algebras,* `https://arxiv.org/abs/2304.01381`

[14] Peskine, C., Szpiro, L. *Liaison des varieties algebriques I,* Invent. Math., 26 (1974), 271-302.

[15] Sam, S., Weyman, J. *Schubert varieties and finite free resolutions of length three,* Proceedings of the AMS Volume 149, Number 5, Pages 1943–1955 (2021)

[16] Weyman, J. *On the structure of resolutions of length 3,* Journal of Algebra 126 No. 1 (1989) 1-33

[17] Weyman, J. *Free resolutions and root systems,* Annales de l'Institute Fourier 68 (3) (2018) 1241-1296

[18] Weyman, J. *Finite free resolutions and root systems,* `https://icerm.brown.edu/topical_workshops/tw-20-frrt/2020icermbasics-revised-last.pdf`