LICCI IDEALS AS AN OBSTRUCTION TO NOETHERIAN BASE RINGS FOR GENERIC FREE RESOLUTIONS

For each length 3 format (f_0, f_1, f_2, f_3), Weyman constructed a generic resolution $\mathbb{F}^{\mathrm{gen}}$ defined over a ring $\widehat{R}_{\mathrm{gen}}$, which is Noetherian exactly when the format corresponds to a Dynkin diagram. Unlike the case of universal objects, the pair $(\widehat{R}_{\mathrm{gen}}, \mathbb{F}^{\mathrm{gen}})$ is not uniquely determined by its genericity property. As such, one might wonder whether there exists a different construction of a generic free resolution that results in a Noetherian base ring for the non-Dynkin cases. The purpose of this note is to explain why this is impossible. We assume $f_0 = 1$, but comment at the end how one might adapt this to $f_0 > 1$.

In what follows, we fix a non-Dynkin format $(1, f_1, f_2, f_3)$. All resolutions considered are assumed to have this format. Loosely put, there exist "arbitrarily complicated" resolutions of licci ideals with the given format, and this poses an obstruction to Noetherianity. We now proceed to make this argument precise.

Lemma 1. *Let S be a $\mathbb{C}$-algebra, U be a $\prod GL(F_i)$-representation in $\widehat{R}_{\mathrm{gen}}$ and let $\overline{U}$ denote the $\mathbb{L}$-representation it generates. Then if M is an S-module resolved by $\mathbb{F}$ and $w\colon \widehat{R}_{\mathrm{gen}} \to S$ specializes $\mathbb{F}^{\mathrm{gen}}$ to $\mathbb{F}$, then the ideal $w(\overline{U})S$ depends only on M and not on the choice of $\mathbb{F}$ or w.*

Proof. Let $w'\colon \widehat{R}_{\mathrm{gen}} \to S$ be another map specializing $\mathbb{F}^{\mathrm{gen}}$ to a resolution $\mathbb{F}'$ of M. To show that $w(\overline{U})S = w'(\overline{U})S$, it suffices to check after localizing at each prime of S, so we reduce at once to the case that S is local (which is the primary case of interest anyway).

In this situation, the resolutions $\mathbb{F}$ and $\mathbb{F}'$ of M must be isomorphic, hence related by the action of $\prod GL(F_i \otimes S)$. Different choices of $w\colon \widehat{R}_{\mathrm{gen}} \to S$ specializing $\mathbb{F}^{\mathrm{gen}}$ to a fixed $\mathbb{F}$ are related by the exponential action of $\mathbb{L} \otimes S$ on $\widehat{R}_{\mathrm{gen}} \otimes S$. As $\overline{U} \otimes S$ is closed under both of these actions, the result follows. ☐

Given $w\colon \widehat{R}_{\mathrm{gen}} \to S$, we write $w^{(1)}$ for its restriction to the representation

$$W(d_1) = F_1 \oplus F_3^* \otimes \bigwedge^3 F_1 \oplus \cdots \subset \widehat{R}_{\mathrm{gen}}$$

If we take U to be the jth graded component of $W(d_1)$, with the convention that F_1 corresponds to $j = 0$, then $\overline{U}$ is the sum of all components up to and including U. We let $I_{\leq j}$ denote the ideal it generates in $\widehat{R}_{\mathrm{gen}}$.

Theorem 2. *Suppose that $(R', \mathbb{F}')$ is a generic pair (over $\mathbb{C}$) for the given non-Dynkin format $(1, f_1, f_2, f_3)$. Then R' is not Noetherian.*

Proof. Fix a map $w'\colon \widehat{R}_{\mathrm{gen}} \to R'$ specializing $\mathbb{F}^{\mathrm{gen}}$ to $\mathbb{F}'$. Then we claim that the following chain of ideals in R' is strictly increasing:

$$w'(I_{\leq 0})R' \subsetneq w'(I_{\leq 1})R' \subsetneq w'(I_{\leq 2})R' \subsetneq \cdots$$

First we prove $w'(I_{\leq 0})R' \subsetneq w'(I_{\leq 1})R'$ for illustrative purposes. Let $S = \mathbb{C}[u_1, u_2, u_3]_{\mathfrak{m}}$ where $\mathfrak{m} = (u_1, u_2, u_3)$. Let $\mathbb{F}$ be a (non-minimal) resolution of $S/(u_1, u_2, u_3)$ of the given format and $\phi\colon R' \to S$ a map specializing $\mathbb{F}'$ to $\mathbb{F}$. Then in particular $\phi w'$ is a map specializing $\mathbb{F}^{\mathrm{gen}}$ to $\mathbb{F}$. Its restriction to $F_3^* \otimes \bigwedge^3 F_1 \subset W(d_1)$ comes from a choice of multiplication on $\mathbb{F}$, and because (u_1, u_2, u_3) is a complete intersection, we know this multiplication to be nonzero mod $\mathfrak{m}$. Thus

$$\phi w'(I_{\le 0})S = (u_1, u_2, u_3) \subsetneq \phi w'(I_{\le 1})S = (1)$$

and consequently $w'(I_{\le 0})R' \subsetneq w'(I_{\le 1})R'$.

Just as how a complete intersection may be used to prove the first inequality, more complicated licci ideals may be used for the remaining ones. For example, taking a type 2 almost complete intersection or a deviation 2 Gorenstein ideal instead of (u_1, u_2, u_3) would show that $w'(I_{\le 1})R' \subsetneq w'(I_{\le 2})R'$. The argument in general is provided in the following lemma (a more detailed and expanded form of which should be put in the forthcoming paper on linkage). $\square$

Lemma 3. *For every positive integer j, there exists a licci ideal I in a local $\mathbb{C}$-algebra $(S, \mathfrak{m}, k)$ such that $w(I_{\le j-1})S \subsetneq w(I_{\le j})S = (1)$ where $w\colon \widehat{R}_{\text{gen}} \to S$ specializes $\mathbb{F}^{\text{gen}}$ to a resolution of S/I.*

The lemma is stated in a way to finish the proof of the theorem, but we really show that for any Noetherian local $\mathbb{C}$-algebra S with depth $S \ge 3$ there is a licci ideal whose resolution admits a choice of $w\colon \widehat{R}_{\text{gen}} \to S$ having the property that $w^{(1)} \otimes k$ is a vector with *any* prescribed extremal weight in $V(\omega_{x_1})$.

Proof. Let v denote a highest weight vector in $V(\omega_{x_1})$, let $\sigma \in W^{P_{x_1}}$ be such that σv is in the j-th graded component of $V(\omega_{x_1})$ (such a σ exists for all $j \ge 0$ because T is not Dynkin!), and let $S = \mathbb{C}[u_1, u_2, u_3]_{\mathfrak{m}}$ where $\mathfrak{m} = (u_1, u_2, u_3)$. Express σ as a product of simple reflections, and lift each reflection to either $GL(F_1 \otimes S) \times GL(F_3 \otimes S)$ (corresponding to $T - \{z_1\}$) or $GL(F_1' \otimes S) \times GL(F_3' \otimes S)$ (corresponding to $T - \{y_1\}$). We obtain a product $g_1 g_2 \cdots g_n$ representing σ in this manner, where $g_i \in GL(F_1 \otimes S) \times GL(F_3 \otimes S)$ if i is even and $g_i \in GL(F_1' \otimes S) \times GL(F_3' \otimes S)$ if i is odd. Let $p\colon S \otimes V(\omega_{x_1}) \to F_1^*$ denote the projection onto the top z_1-graded piece, $p'\colon S \otimes V(\omega_{x_1}) \to F_1'^*$ the projection onto the top y_1-graded piece, and $p_{\text{link}}\colon S \otimes V(\omega_{x_1}) \to S^3$ the projection onto the overlap. By adding a general matrix with entries in $\mathfrak{m}$ to each g_i, we can arrange for $p_{\text{link}}(g_k g_{k+1} \cdots g_n v)$ to be a regular sequence in S (or to contain a unit) for all k. Then, writing $I_1(-)$ to denote the ideal generated by the entries of $(-)$,

$$\begin{aligned} I_1 p(g_1 g_2 g_3 \cdots g_n v) &\sim I_1 p'(g_1 g_2 g_3 \cdots g_n v) && \text{by the reg. seq. } p_{\text{link}}(g_1 g_2 g_3 \cdots g_n v) \\ I_1 p'(g_1 g_2 g_3 \cdots g_n v) = I_1 p'(g_2 g_3 \cdots g_n v) &\sim I_1 p(g_2 g_3 \cdots g_n v) && \text{by the reg. seq. } p_{\text{link}}(g_2 g_3 \cdots g_n v) \\ I_1 p(g_2 g_3 \cdots g_n v) = I_1 p(g_3 \cdots g_n v) &\sim \cdots \end{aligned}$$

so the ideal $I = I_1 p(g_1 g_2 \cdots g_n v)$ is licci. A map $w\colon \widehat{R}_{\text{gen}} \to S$ specializing $\mathbb{F}^{\text{gen}}$ to S can be obtained by taking the "standard split structure" $w_{\text{ssc}}\colon \widehat{R}_{\text{gen}} \to \mathbb{C}$, viewing it as a map $\widehat{R}_{\text{gen}} \otimes S \to S$, and then precomposing with the action of $(g_1 g_2 \cdots g_n)^{-1}$ on $\widehat{R}_{\text{gen}} \otimes S$. The map w_{ssc} restricted to $W(d_1)$ is just (the dual of) v, so by design our $w^{(1)} \otimes k$ is σv. In particular, $w(I_{\le j-1}) \otimes k = 0$ and $w(I_{\le j}) \otimes k \ne 0$. $\square$

For this lemma, one can alternatively let S be the local ring of the finite-dimensional Schubert cell $C_\sigma = B^+ \sigma v$ at σv, and let $I \subset S$ be the ideal of $X^w \cap C_\sigma$ at that point, where $w = s_{z_1} s_u s_{x_1} \in W^{P_{x_1}}$ and X^w is the codimension 3 Schubert variety $\overline{B^- w v}$. A similar construction to this could be used to prove Theorem 2 for non-Dynkin module formats as well, controlling the location of a unit appearing in $W(a_2)$ instead of $W(d_1)$. But we need to locate references for some statements regarding Schubert ind-varieties...