# Weyman's generic ring for free resolutions of length three

L. Guerrieri[1] X. Ni[2] J. Weyman[3]

[1]Jagiellonian University

[2]UC Berkeley

[3]Jagiellonian University

Joint Mathematics Meetings, January 2024

# Background

Throughout, $I$ will denote an ideal in a regular local $\mathbb{C}$-algebra $(S, \mathfrak{m}, k)$.

# Background

Throughout, $I$ will denote an ideal in a regular local $\mathbb{C}$-algebra $(S, \mathfrak{m}, k)$.

The module $S/I$ has a *finite free resolution*, i.e. an acyclic complex

$$\mathbb{F}\colon 0 \to F_c \xrightarrow{d_c} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} S$$

where $F_i = S^{f_i}$ is in homological degree $i$, $F_0 = S$, and $H_0(\mathbb{F}) = S/I$. We call $(f_0, f_1, \ldots, f_c)$ the *format* of $\mathbb{F}$.

# Background

Throughout, $I$ will denote an ideal in a regular local $\mathbb{C}$-algebra $(S, \mathfrak{m}, k)$.
The module $S/I$ has a *finite free resolution*, i.e. an acyclic complex

$$\mathbb{F} \colon 0 \to F_c \xrightarrow{d_c} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} S$$

where $F_i = S^{f_i}$ is in homological degree $i$, $F_0 = S$, and $H_0(\mathbb{F}) = S/I$. We call $(f_0, f_1, \ldots, f_c)$ the *format* of $\mathbb{F}$.
If $d_i \otimes k = 0$ for all $i$, then $\mathbb{F}$ is *minimal*, and $b_i := f_i$ are the (ordinary) *Betti numbers* of $S/I$. The *projective dimension* $\operatorname{pdim} S/I$ is then equal to $c$.

# Background

The ideal $I \subset S$ is called *perfect* if

$$\operatorname{pdim} S/I = \operatorname{grade} I := \min\{i : \operatorname{Ext}^i(S/I, S) \neq 0\}.$$

Since $S$ is regular, this is equivalent to $S/I$ being Cohen-Macaulay.

# Background

The ideal $I \subset S$ is called *perfect* if

$$\operatorname{pdim} S/I = \operatorname{grade} I := \min\{i : \operatorname{Ext}^i(S/I, S) \neq 0\}.$$

Since $S$ is regular, this is equivalent to $S/I$ being Cohen-Macaulay.
Minimal free resolution of $S/I$:

$$\mathbb{F}\colon 0 \to F_c \xrightarrow{d_c} F_{c-1} \xrightarrow{d_{c-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} S$$

Dual:

$$\mathbb{F}^*\colon 0 \to S \xrightarrow{d_1^*} F_1^* \xrightarrow{d_2^*} \cdots \xrightarrow{d_{c-1}^*} F_{c-1}^* \xrightarrow{d_c^*} F_c^*$$

# Background

The ideal $I \subset S$ is called *perfect* if

$$\operatorname{pdim} S/I = \operatorname{grade} I := \min\{i : \operatorname{Ext}^i(S/I, S) \neq 0\}.$$

Since $S$ is regular, this is equivalent to $S/I$ being Cohen-Macaulay.
Minimal free resolution of $S/I$:

$$\mathbb{F}\colon 0 \to F_c \xrightarrow{d_c} F_{c-1} \xrightarrow{d_{c-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} S$$

Dual:

$$\mathbb{F}^*\colon 0 \to S \xrightarrow{d_1^*} F_1^* \xrightarrow{d_2^*} \cdots \xrightarrow{d_{c-1}^*} F_{c-1}^* \xrightarrow{d_c^*} F_c^*$$

$$I \text{ is perfect} \iff c = \min\{i : \operatorname{Ext}^i(S/I, S) \neq 0\}$$

# Background

The ideal $I \subset S$ is called *perfect* if

$$\operatorname{pdim} S/I = \operatorname{grade} I := \min\{i : \operatorname{Ext}^i(S/I, S) \neq 0\}.$$

Since $S$ is regular, this is equivalent to $S/I$ being Cohen-Macaulay. Minimal free resolution of $S/I$:

$$\mathbb{F} \colon 0 \to F_c \xrightarrow{d_c} F_{c-1} \xrightarrow{d_{c-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} S$$

Dual:

$$\mathbb{F}^* \colon 0 \to S \xrightarrow{d_1^*} F_1^* \xrightarrow{d_2^*} \cdots \xrightarrow{d_{c-1}^*} F_{c-1}^* \xrightarrow{d_c^*} F_c^*$$

$$\begin{aligned} I \text{ is perfect } &\iff c = \min\{i : \operatorname{Ext}^i(S/I, S) \neq 0\} \\ &\iff \mathbb{F}^* \text{ is acyclic.} \end{aligned}$$

In this case, $\mathbb{F}^*$ resolves the canonical module $\operatorname{Ext}^c(S/I, S)$. The minimal number of generators of this module is $b_c$, which is called the (Cohen-Macaulay) *type* of $S/I$.

# Perfect ideals with $c = 2$

## Theorem (Hilbert-Burch)

*Every free resolution of the form*

$$0 \to F_2 \to F_1 \to S$$

*is uniquely a specialization of*

$$0 \to T^{n-1} \xrightarrow{d_2} T^n \xrightarrow{d_1} T$$

*for some $n$, where $T = \mathbb{C}[x_{ij}, y]$, $d_2$ is a generic matrix in the $x_{ij}$, and $d_1 = y(\bigwedge^{n-1} d_2^*)$.*

# Perfect ideals with $c = 2$

## Theorem (Hilbert-Burch)

*Every free resolution of the form*

$$0 \to F_2 \to F_1 \to S$$

*is uniquely a specialization of*

$$0 \to T^{n-1} \xrightarrow{d_2} T^n \xrightarrow{d_1} T$$

*for some $n$, where $T = \mathbb{C}[x_{ij}, y]$, $d_2$ is a generic matrix in the $x_{ij}$, and $d_1 = y(\bigwedge^{n-1} d_2^*)$.*

## Corollary

*Every grade 2 perfect ideal is generated by the $(n-1) \times (n-1)$ minors of a $(n-1) \times n$ matrix.*

# Perfect ideals with $c = 3$

Henceforth we restrict our attention to $c = 3$.

# Perfect ideals with $c = 3$ ... and type 1

Henceforth we restrict our attention to $c = 3$. The ring $R = S/I$ is *Gorenstein* when $I \subset S$ is perfect and $b_c = 1$. We refer to $I$ itself as a Gorenstein ideal.

# Perfect ideals with $c = 3$ ... and type 1

Henceforth we restrict our attention to $c = 3$. The ring $R = S/I$ is *Gorenstein* when $I \subset S$ is perfect and $b_c = 1$. We refer to $I$ itself as a Gorenstein ideal.

In this case, the minimal free resolution $\mathbb{F}$ of $S/I$ looks like

$$\mathbb{F}\colon 0 \to S \to S^n \to S^n \to S.$$

# Perfect ideals with $c = 3$ ... and type 1

Henceforth we restrict our attention to $c = 3$. The ring $R = S/I$ is *Gorenstein* when $I \subset S$ is perfect and $b_c = 1$. We refer to $I$ itself as a Gorenstein ideal.
In this case, the minimal free resolution $\mathbb{F}$ of $S/I$ looks like

$$\mathbb{F}\colon 0 \to S \to S^n \to S^n \to S.$$

## Theorem (Watanabe)

*$n$ is odd.*

# A story about a well-known case

**Theorem (Watanabe)**

*$n$ is odd.*

**Proof.**

Starting from $I$, produce another grade 3 Gorenstein ideal with $n-2$ minimal generators. Repeat this until it terminates at a complete intersection, which is generated by 3 elements. So the original number $n$ must be odd. $\square$

# A story about a well-known case

**Theorem (Watanabe)**

*$n$ is odd.*

**Proof.**

Starting from $I$, produce another grade 3 Gorenstein ideal with $n-2$ minimal generators. Repeat this until it terminates at a complete intersection, which is generated by 3 elements. So the original number $n$ must be odd. $\square$

Watanabe's procedure is a special case of what is now known as *linkage*. (Details omitted—not the focus of this talk.)

$$I = I_0 \sim I_1 \sim I_2 \sim I_3 \sim \cdots \sim K = \text{complete intersection}$$

Recurring theme in various later works: translate properties from $K$, which is well-understood, back to the original ideal $I$.

# A story about a well-known case

### Theorem (Buchsbaum-Eisenbud)

*Fix an odd $n$. Every free resolution of the form*

$$0 \to S \to S^n \to S^n \to S$$

*with acyclic dual is isomorphic to a specialization of*

$$0 \to T \to T^n \to T^n \to T$$

*where $T = \mathbb{C}[x_{ij}]$, $d_2$ is a generic skew matrix in the $x_{ij}$, and $d_1, d_3$ have entries given by the $(n-1) \times (n-1)$ pfaffians of $d_2$.*

# A story about a well-known case

### Theorem (Buchsbaum-Eisenbud)

*Fix an odd $n$. Every free resolution of the form*

$$0 \to S \to S^n \to S^n \to S$$

*with acyclic dual is isomorphic to a specialization of*

$$0 \to T \to T^n \to T^n \to T$$

*where $T = \mathbb{C}[x_{ij}]$, $d_2$ is a generic skew matrix in the $x_{ij}$, and $d_1, d_3$ have entries given by the $(n-1) \times (n-1)$ pfaffians of $d_2$.*

Note that this is not quite analogous to Hilbert-Burch:

- It only concerns resolutions with acyclic dual
- The specialization is only up to isomorphism, and not unique

# Universal free resolutions

Hilbert-Burch gives the *universal example* of a free resolution with format $(1, n, n-1)$. What's the universal example for $(1, n, n, 1)$?

1

## Universal free resolutions

Hilbert-Burch gives the *universal example* of a free resolution with format $(1, n, n-1)$. What's the universal example for $(1, n, n, 1)$?

**Theorem (Bruns)**

*There isn't one.*

# ~~Universal~~ Generic free resolutions

Hilbert-Burch gives the *universal example* of a free resolution with format $(1, n, n-1)$. What's the universal example for $(1, n, n, 1)$?

### Theorem (Bruns)

*There isn't one. But if we drop the requirement that the specialization be unique, so that the example is "generic" instead of "universal," then it always*[1] *exists.*

[1] for arbitrary formats of any length, not just $(1, n, n, 1)$.

# Weyman's generic ring

For each format $(f_0, f_1, f_2, f_3)$ of length 3, Weyman constructed a generic ring $\widehat{R}_{\mathrm{gen}}$ and a generic resolution

$$\mathbb{F}^{\mathrm{gen}}\colon 0 \to \widehat{R}_{\mathrm{gen}}^{f_3} \to \widehat{R}_{\mathrm{gen}}^{f_2} \to \widehat{R}_{\mathrm{gen}}^{f_1} \to \widehat{R}_{\mathrm{gen}}^{f_0}$$

specializing to any resolution $\mathbb{F}$ of the same format, carefully keeping track of the non-uniqueness of $\widehat{R}_{\mathrm{gen}} \to S$.

# Weyman's generic ring

For each format $(f_0, f_1, f_2, f_3)$ of length 3, Weyman constructed a generic ring $\widehat{R}_{\mathrm{gen}}$ and a generic resolution

$$\mathbb{F}^{\mathrm{gen}} \colon 0 \to \widehat{R}_{\mathrm{gen}}^{f_3} \to \widehat{R}_{\mathrm{gen}}^{f_2} \to \widehat{R}_{\mathrm{gen}}^{f_1} \to \widehat{R}_{\mathrm{gen}}^{f_0}$$

specializing to any resolution $\mathbb{F}$ of the same format, carefully keeping track of the non-uniqueness of $\widehat{R}_{\mathrm{gen}} \to S$.

Continuing our example from before, let $(f_0, f_1, f_2, f_3) = (1, n, n, 1)$. The resolution $\mathbb{F}$ does not uniquely determine a map $\widehat{R}_{\mathrm{gen}} \to S$ specializing $\mathbb{F}^{\mathrm{gen}}$ to $\mathbb{F}$; some additional data is required.

# Weyman's generic ring

For each format $(f_0, f_1, f_2, f_3)$ of length 3, Weyman constructed a generic ring $\widehat{R}_{\mathrm{gen}}$ and a generic resolution

$$\mathbb{F}^{\mathrm{gen}} \colon 0 \to \widehat{R}_{\mathrm{gen}}^{f_3} \to \widehat{R}_{\mathrm{gen}}^{f_2} \to \widehat{R}_{\mathrm{gen}}^{f_1} \to \widehat{R}_{\mathrm{gen}}^{f_0}$$

specializing to any resolution $\mathbb{F}$ of the same format, carefully keeping track of the non-uniqueness of $\widehat{R}_{\mathrm{gen}} \to S$.
Continuing our example from before, let $(f_0, f_1, f_2, f_3) = (1, n, n, 1)$.
The resolution $\mathbb{F}$ does not uniquely determine a map $\widehat{R}_{\mathrm{gen}} \to S$ specializing $\mathbb{F}^{\mathrm{gen}}$ to $\mathbb{F}$; some additional data is required.
For this particular format, the additional data is a choice of multiplicative structure on $\mathbb{F}$, making it into a CDGA:

$$\begin{tikzcd}
0 \arrow[r] & F_3 \arrow[r, "d_3"] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & S \\
 & & & \bigwedge^2 F_1 \arrow[ul, dashed] \arrow[u, "d_1 \otimes 1 - 1 \otimes d_1"'] &
\end{tikzcd}$$

# Weyman's generic ring

For each format $(f_0, f_1, f_2, f_3)$ of length 3, Weyman constructed a generic ring $\widehat{R}_{\mathrm{gen}}$ and a generic resolution

$$\mathbb{F}^{\mathrm{gen}}\colon 0 \to \widehat{R}_{\mathrm{gen}}^{f_3} \to \widehat{R}_{\mathrm{gen}}^{f_2} \to \widehat{R}_{\mathrm{gen}}^{f_1} \to \widehat{R}_{\mathrm{gen}}^{f_0}$$

specializing to any resolution $\mathbb{F}$ of the same format, carefully keeping track of the non-uniqueness of $\widehat{R}_{\mathrm{gen}} \to S$.

Continuing our example from before, let $(f_0, f_1, f_2, f_3) = (1, n, n, 1)$.

The resolution $\mathbb{F}$ does not uniquely determine a map $\widehat{R}_{\mathrm{gen}} \to S$ specializing $\mathbb{F}^{\mathrm{gen}}$ to $\mathbb{F}$; some additional data is required.

For this particular format, the additional data is a choice of multiplicative structure on $\mathbb{F}$, making it into a CDGA:

$$\begin{tikzcd}
0 \arrow[r] & F_3 \arrow[r, "d_3"] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & S \\
& & & \bigwedge^2 F_1 \arrow[u, "d_1\otimes 1-1\otimes d_1"'] \arrow[ul, dashed] \arrow[ull, red] &
\end{tikzcd}$$

Note that the non-uniqueness of this choice is $F_3 \otimes \bigwedge^2 F_1^*$.

# The group action

The group $\prod GL(F_i)$ acts on

$$\{\text{free resolutions } \mathbb{F} \text{ of format } (1, n, n, 1)\}$$

# The group action

The group $\prod GL(F_i)$ acts on

$$\left\{\begin{array}{c}\text{free resolutions } \mathbb{F} \text{ of format } (1,n,n,1)\\ \text{together with choice of multiplicative structure}\end{array}\right\}$$

and thus on $\widehat{R}_{\mathrm{gen}}$.

# The group action

The (additive) group $F_3 \otimes \bigwedge^2 F_1^*$ acts on

$$\left\{ \begin{array}{c} \text{free resolutions } \mathbb{F} \text{ of format } (1, n, n, 1) \\ \text{together with choice of multiplicative structure} \end{array} \right\}$$

and thus on $\widehat{R}_{\mathrm{gen}}$.

$$\begin{tikzcd}
0 \arrow[r] & F_3 \arrow[r, "d_3"] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & S \\
 & & & \bigwedge^2 F_1 \arrow[u, "d_1 \otimes 1 - 1 \otimes d_1"'] \arrow[ul, dashed] \arrow[ull, "g"]
\end{tikzcd}$$

The action of $g \in F_3 \otimes \bigwedge^2 F_1^*$ fixes $\mathbb{F}$ and adds $d_3 g$ to the choice of multiplication $\bigwedge^2 F_1 \to F_2$.

# The group action

The semidirect product $(F_3 \otimes \bigwedge^2 F_1^*) \rtimes \prod GL(F_i)$ acts on

$$\left\{ \begin{array}{c} \text{free resolutions } \mathbb{F} \text{ of format } (1, n, n, 1) \\ \text{together with choice of multiplicative structure} \end{array} \right\}$$

and thus on $\widehat{R}_{\mathrm{gen}}$.

This group turns out to be closely related to a parabolic subgroup of an even larger group.

# A Lie algebra acting on $\widehat{R}_{\mathrm{gen}}$

One of the key results from Weyman's 2018 paper, presented here for the case $(1, n, n, 1)$.

$$F_3^* \otimes \bigwedge^2 F_1$$

$$\mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{sl}(F_1) \times \mathfrak{sl}(F_3) \times \mathbb{C}$$

$$F_3 \otimes \bigwedge^2 F_1^*$$

# A Lie algebra acting on $\widehat{R}_{\mathrm{gen}}$

One of the key results from Weyman's 2018 paper, presented here for the case $(1, n, n, 1)$.

$$\begin{array}{c} \\ \mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \\ \\ \end{array} \begin{array}{c} F_3^* \otimes \bigwedge^2 F_1 \\ \mathfrak{sl}(F_1) \times \mathfrak{sl}(F_3) \times \mathbb{C} \\ F_3 \otimes \bigwedge^2 F_1^* \end{array}$$

- Right part: the Lie algebra $D_n = \mathfrak{so}_{2n}$, displayed with grading induced by vertex $n$.

$$\begin{array}{ccccccccccc} (n-1) & - & (n-2) & - & (n-3) & - & \cdots & - & 2 & - & 1 \\ & & | & & & & & & & & \\ & & n & & & & & & & & \end{array}$$

# A Lie algebra acting on $\widehat{R}_{\mathrm{gen}}$

One of the key results from Weyman's 2018 paper, presented here for the case $(1, n, n, 1)$.

$$\begin{array}{c} F_3^* \otimes \bigwedge^2 F_1 \\ \mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{sl}(F_1) \times \mathfrak{sl}(F_3) \times \mathbb{C} \\ F_3 \otimes \bigwedge^2 F_1^* \end{array}$$

- Right part: the Lie algebra $D_n = \mathfrak{so}_{2n}$, displayed with grading induced by vertex $n$.
- Left part: type $A_{n-1}$ subalgebra of $D_n = \mathfrak{so}_{2n}$ corresponding to $D_n - \{(n-1)\}$.

$$\begin{array}{ccccccccccccc} (n-1) & - & (n-2) & - & (n-3) & - & \cdots & - & 2 & - & 1 \\ & & | & & & & & & & & \\ & & n & & & & & & & & \end{array}$$

# A new perspective on classical results

So free resolutions of format $(1, n, n, 1)$ have "something" to do with the the Dynkin diagram $D_n$, especially the two nodes $n - 1$ and $n$.

# A new perspective on classical results

So free resolutions of format $(1, n, n, 1)$ have "something" to do with the the Dynkin diagram $D_n$, especially the two nodes $n - 1$ and $n$.

Recall:

### Theorem (Buchsbaum-Eisenbud)

*If $I$ is a grade 3 Gorenstein ideal, then $S/I$ has a minimal free resolution of the form*

$$0 \to S \to S^n \to S^n \to S$$

*where $d_2$ is a skew matrix and $d_1, d_3$ have entries given by the $(n-1) \times (n-1)$ pfaffians of $d_2$.*

# A new perspective on classical results

So free resolutions of format $(1, n, n, 1)$ have "something" to do with the the Dynkin diagram $D_n$, especially the two nodes $n-1$ and $n$.
Recall:

### Theorem (Buchsbaum-Eisenbud)

*If $I$ is a grade 3 Gorenstein ideal, then $S/I$ has a minimal free resolution of the form*

$$0 \to S \to S^n \to S^n \to S$$

*where $d_2$ is a skew matrix and $d_1, d_3$ have entries given by the $(n-1) \times (n-1)$ pfaffians of $d_2$.*

An arbitrary minimal free resolution of $S/I$ certainly won't have this form. The skew matrix only appears after $F_2$ is identified with $F_1^*$ using *a choice of multiplication* $F_1 \otimes F_2 \to F_3$.

# A new perspective on classical results

An $n \times (2n)$ block matrix:

$$\begin{array}{cc} & \begin{array}{cc} F_1^* & \qquad F_3^* \otimes F_1 \end{array} \\ F_2^* & \begin{bmatrix} \begin{array}{c}\text{original differential} \\ d_2^* \end{array} & \text{multiplication} \end{bmatrix} \end{array}$$

# A new perspective on classical results

An $n \times (2n)$ block matrix:

$$\begin{array}{cc} & \begin{array}{cc} F_1^* & F_3^* \otimes F_1 \end{array} \\ F_2^* & \begin{bmatrix} \begin{array}{c}\text{original differential} \\ d_2^*\end{array} & \text{multiplication} \end{bmatrix} \end{array}$$

"identify $F_2 \cong F_1^*$" $=$ do row operations to make second block the identity:

$$\begin{array}{cc} & \begin{array}{cc} F_1^* & F_3^* \otimes F_1 \end{array} \\ F_2^* & \begin{bmatrix} \begin{array}{c}\text{new differential} \\ d_2^*\end{array} & I_n \end{bmatrix} \end{array}$$

Buchsbaum-Eisenbud: the left block is now a skew matrix.

# A new perspective on classical results

An $n \times (2n)$ block matrix:

$$\begin{array}{cc} & \begin{array}{cc} F_1^* & F_3^* \otimes F_1 \end{array} \\ F_2^* & \begin{bmatrix} \begin{array}{c}\text{original differential} \\ d_2^* \end{array} & \text{multiplication} \end{bmatrix} \end{array}$$

"identify $F_2 \cong F_1^*$" $=$ do row operations to make second block the identity:

$$\begin{array}{cc} & \begin{array}{cc} F_1^* & F_3^* \otimes F_1 \end{array} \\ F_2^* & \begin{bmatrix} \begin{array}{c}\text{new differential} \\ d_2^* \end{array} & I_n \end{bmatrix} \end{array}$$

Buchsbaum-Eisenbud: the left block is now a skew matrix. In other words, this determines a $S$-point of the *orthogonal Grassmannian* $OG(n, 2n)$ of isotropic $n$-planes in $F_1 \oplus F_3 \otimes F_1^*$.

# A new perspective on classical results

$$\begin{array}{cc} & F_1^* \qquad\qquad F_3^* \otimes F_1 \\ F_2^* & \begin{bmatrix} \text{skew differential} & \\ d_2^* & I_n \end{bmatrix} \end{array}$$

The orthogonal Grassmannian $OG(n, 2n)$ is the homogeneous space $D_n/P_{n-1}$.

# A new perspective on classical results

$$\begin{array}{cc} & \begin{array}{cc} F_1^* & F_3^* \otimes F_1 \end{array} \\ F_2^* & \begin{bmatrix} \text{skew differential } d_2^* & I_n \end{bmatrix} \end{array}$$

The orthogonal Grassmannian $OG(n, 2n)$ is the homogeneous space $D_n/P_{n-1}$. Moreover, the submaximal pfaffians of $d_2$ correspond to the Plücker coordinates cutting out the codimension 3 Schubert variety $X^{s_n s_{n-2} s_{n-1}}$ complementary to the open $P_n$-orbit in $D_n/P_{n-1}$.

# A new perspective on classical results

$$\begin{array}{cc} & \begin{array}{cc} F_1^* & F_3^* \otimes F_1 \end{array} \\ F_2^* & \begin{bmatrix} \text{skew differential} & \\ d_2^* & I_n \end{bmatrix} \end{array}$$

The orthogonal Grassmannian $OG(n, 2n)$ is the homogeneous space $D_n/P_{n-1}$. Moreover, the submaximal pfaffians of $d_2$ correspond to the Plücker coordinates cutting out the codimension 3 Schubert variety $X^{s_n s_{n-2} s_{n-1}}$ complementary to the open $P_n$-orbit in $D_n/P_{n-1}$.

This Schubert variety itself decomposes into a number of $P_n$-orbits. The local ring of $X^{s_n s_{n-2} s_{n-1}}$ at some point $p$ depends on the orbit containing $p$.

# A new perspective on classical results

$$\begin{array}{cc} & F_1^* & F_3^* \otimes F_1 \\ F_2^* & \left[\begin{array}{c} \text{skew differential} \\ d_2^* \end{array}\right. & \left.\begin{array}{c} I_n \end{array}\right] \end{array}$$

The orthogonal Grassmannian $OG(n, 2n)$ is the homogeneous space $D_n/P_{n-1}$. Moreover, the submaximal pfaffians of $d_2$ correspond to the Plücker coordinates cutting out the codimension 3 Schubert variety $X^{s_n s_{n-2} s_{n-1}}$ complementary to the open $P_n$-orbit in $D_n/P_{n-1}$.
This Schubert variety itself decomposes into a number of $P_n$-orbits. The local ring of $X^{s_n s_{n-2} s_{n-1}}$ at some point $p$ depends on the orbit containing $p$. It can have Betti numbers...

$(1, n, n, 1)$
smallest orbit

# A new perspective on classical results

$$\begin{array}{ccc} & F_1^* & F_3^* \otimes F_1 \\ F_2^* & \left[\begin{array}{c} \text{skew differential} \\ d_2^* \end{array}\right. & \left. \begin{array}{c} I_n \end{array} \right] \end{array}$$

The orthogonal Grassmannian $OG(n, 2n)$ is the homogeneous space $D_n/P_{n-1}$. Moreover, the submaximal pfaffians of $d_2$ correspond to the Plücker coordinates cutting out the codimension 3 Schubert variety $X^{s_n s_{n-2} s_{n-1}}$ complementary to the open $P_n$-orbit in $D_n/P_{n-1}$.
This Schubert variety itself decomposes into a number of $P_n$-orbits. The local ring of $X^{s_n s_{n-2} s_{n-1}}$ at some point $p$ depends on the orbit containing $p$. It can have Betti numbers...

$(1, 3, 3, 1)$
largest orbit

$(1, n, n, 1)$
smallest orbit

# A new perspective on classical results

$$
\begin{array}{ccc}
 & F_1^* & F_3^* \otimes F_1 \\
F_2^* & \begin{bmatrix} \text{skew differential} \\ d_2^* \end{bmatrix.} & \begin{bmatrix.} I_n \end{bmatrix}
\end{array}
$$

The orthogonal Grassmannian $OG(n, 2n)$ is the homogeneous space $D_n/P_{n-1}$. Moreover, the submaximal pfaffians of $d_2$ correspond to the Plücker coordinates cutting out the codimension 3 Schubert variety $X^{s_n s_{n-2} s_{n-1}}$ complementary to the open $P_n$-orbit in $D_n/P_{n-1}$.
This Schubert variety itself decomposes into a number of $P_n$-orbits. The local ring of $X^{s_n s_{n-2} s_{n-1}}$ at some point $p$ depends on the orbit containing $p$. It can have Betti numbers...

$$
\underset{\text{largest orbit}}{(1,3,3,1)} \quad (1,5,5,1) \quad \cdots \quad (1, n-2, n-2, 1) \quad \underset{\text{smallest orbit}}{(1,n,n,1)}
$$

# A new perspective on classical results

format $(1, n, n, 1)$ $\qquad$ $D_n = \mathfrak{so}_{2n},\ n-1,\ n$

$$
\begin{array}{ccccccccccccc}
(n-1) & - & (n-2) & - & (n-3) & - & \cdots & - & 2 & - & 1 \\
 & & | & & & & & & & & \\
 & & n & & & & & & & &
\end{array}
$$

# A new perspective on classical results

format $(1, n, n, 1)$ $\qquad$ $D_n = \mathfrak{so}_{2n},\ n-1,\ n$

$$OG(n, 2n) = D_n / P_{n-1}$$

$$\begin{array}{ccccccccccc} (n-1) & - & (n-2) & - & (n-3) & - & \cdots & - & 2 & - & 1 \\ & & | & & & & & & & & \\ & & n & & & & & & & & \end{array}$$

# A new perspective on classical results

| | |
|---|---|
| format $(1, n, n, 1)$ | $D_n = \mathfrak{so}_{2n},\ n-1,\ n$ |
| | $OG(n, 2n) = D_n/P_{n-1}$ |
| Buchsbaum-Eisenbud for grade 3 Gorenstein ideals | local rings of $X^{s_n s_{n-2} s_{n-1}} \subset D_n/P_{n-1}$ |

$$\begin{array}{ccccccccccc} (n-1) & - & (n-2) & - & (n-3) & - & \cdots & - & 2 & - & 1 \\ & & | & & & & & & & & \\ & & n & & & & & & & & \end{array}$$

# A new perspective on classical results

format $(1, n, n, 1)$ | $D_n = \mathfrak{so}_{2n}$, $n-1$, $n$

$OG(n, 2n) = D_n / P_{n-1}$

Buchsbaum-Eisenbud for grade 3 Gorenstein ideals | local rings of $X^{s_n s_{n-2} s_{n-1}} \subset D_n / P_{n-1}$

format $(1, 4, n, n-3)$ | $D_n = \mathfrak{so}_{2n}$, $n-1$, $n-3$

$$\begin{array}{ccccccccccccc} (n-1) & - & (n-2) & - & (n-3) & - & \cdots & - & 2 & - & 1 \\ & & | & & & & & & & & \\ & & n & & & & & & & & \end{array}$$

# A new perspective on classical results

format $(1, n, n, 1)$ — $D_n = \mathfrak{so}_{2n}$, $n-1$, $n$

$OG(n, 2n) = D_n/P_{n-1}$

Buchsbaum-Eisenbud for grade 3 Gorenstein ideals — local rings of $X^{s_n s_{n-2} s_{n-1}} \subset D_n/P_{n-1}$

$OG(n, 2n) = D_n/P_{n-1}$

format $(1, 4, n, n-3)$ — $D_n = \mathfrak{so}_{2n}$, $n-1$, $n-3$

$$\begin{array}{ccccccccccccc} (n-1) & - & (n-2) & - & (n-3) & - & \cdots & - & 2 & - & 1 \\ & & | & & & & & & & & \\ & & n & & & & & & & & \end{array}$$

# A new perspective on classical results

| | |
|---|---|
| format $(1, n, n, 1)$ | $D_n = \mathfrak{so}_{2n}$, $n-1$, $n$ |
| | $OG(n, 2n) = D_n/P_{n-1}$ |
| Buchsbaum-Eisenbud for grade 3 Gorenstein ideals | local rings of $X^{s_n s_{n-2} s_{n-1}} \subset D_n/P_{n-1}$ |
| Buchsbaum-Eisenbud for grade 3 a.c.i. ideals | local rings of $X^{s_{n-3} s_{n-2} s_{n-1}} \subset D_n/P_{n-1}$ |
| | $OG(n, 2n) = D_n/P_{n-1}$ |
| format $(1, 4, n, n-3)$ | $D_n = \mathfrak{so}_{2n}$, $n-1$, $n-3$ |

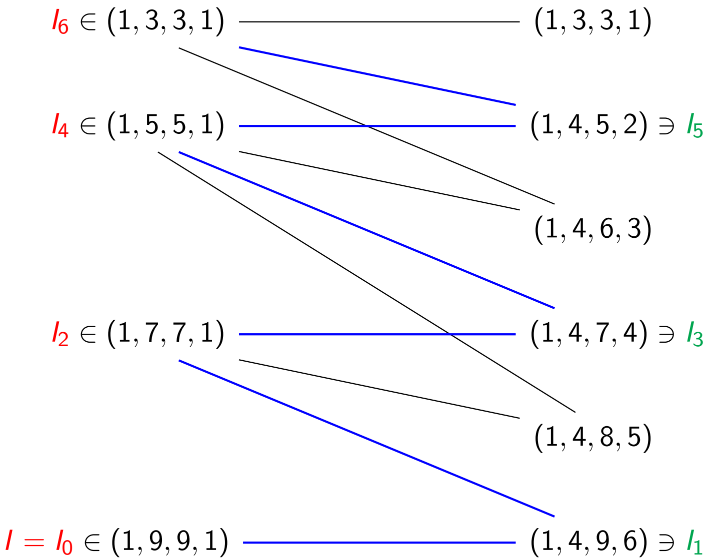
$$\begin{array}{ccccccccccccc} (n-1) & - & (n-2) & - & (n-3) & - & \cdots & - & 2 & - & 1 \\ & & | & & & & & & & & \\ & & n & & & & & & & & \end{array}$$

# A new perspective on classical results

| | |
|---|---|
| format $(1, n, n, 1)$ | $D_n = \mathfrak{so}_{2n},\ n-1,\ n$ |
| | $OG(n, 2n) = D_n/P_{n-1}$ |
| Buchsbaum-Eisenbud for grade 3 Gorenstein ideals | local rings of $X^{s_n s_{n-2} s_{n-1}} \subset D_n/P_{n-1}$ |
| Watanabe (linkage) | geometric linkage of $X^{s_n s_{n-2} s_{n-1}}$ and $X^{s_{n-3} s_{n-2} s_{n-1}}$ |
| Buchsbaum-Eisenbud for grade 3 a.c.i. ideals | local rings of $X^{s_{n-3} s_{n-2} s_{n-1}} \subset D_n/P_{n-1}$ |
| | $OG(n, 2n) = D_n/P_{n-1}$ |
| format $(1, 4, n, n-3)$ | $D_n = \mathfrak{so}_{2n},\ n-1,\ n-3$ |

$$\begin{array}{ccccccccccccc} (n-1) & - & (n-2) & - & (n-3) & - & \cdots & - & 2 & - & 1 \\ & & | & & & & & & & & \\ & & n & & & & & & & & \end{array}$$

# Incidence of non-maximal $P_n$ and $P_{n-3}$ orbits in $D_n/P_{n-1}$



# From a new perspective on classical results...

| | |
|---|---|
| format $(1, n, n, 1)$ | $D_n \ni n-1, n$ |
| structure theory of grade 3 Gorenstein ideals | local rings of $X^{s_n s_{n-2} s_{n-1}} \subset D_n / P_{n-1}$ |
| linkage | geometric linkage of $X^{s_n s_{n-2} s_{n-1}}$ and $X^{s_{n-3} s_{n-2} s_{n-1}}$ |
| structure theory of grade 3 a.c.i. ideals | local rings of $X^{s_{n-3} s_{n-2} s_{n-1}} \subset D_n / P_{n-1}$ |
| format $(1, 4, n, n-3)$ | $D_n \ni n-1, n-3$ |

$$
\begin{array}{ccccccccccc}
(n-1) & - & (n-2) & - & (n-3) & - & \cdots & - & 2 & - & 1 \\
 & & | & & & & & & & & \\
 & & n & & & & & & & &
\end{array}
$$

# ...to new cases

| | |
|---|---|
| format $(1,7,8,2)$ | $E_8 \ni 2, 3$ |
| structure theory of perfect ideals up to $(1,7,8,2)$ | local rings of $X^{s_3 s_4 s_2} \subset E_8/P_2$ |
| linkage | geometric linkage of $X^{s_3 s_4 s_2}$ and $X^{s_5 s_4 s_2}$ |
| structure theory of perfect ideals up to $(1,5,8,4)$ | local rings of $X^{s_5 s_4 s_2} \subset E_8/P_2$ |
| format $(1,5,8,4)$ | $E_8 \ni 2, 5$ |

$$\begin{array}{ccccccccccc} 2 & - & 4 & - & 5 & - & 6 & - & 7 & - & 8 \\ & & | & & & & & & & & \\ & & 3 & & & & & & & & \\ & & | & & & & & & & & \\ & & 1 & & & & & & & & \end{array}$$

# ...to new cases

Table: Number of families of perfect ideals $I$ such that $S/I$ has Betti numbers $(1, 3+d, 2+d+t, t)$

| | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ | $\cdots$ |
|---|---|---|---|---|---|---|
| $d=0$ | 1 | 0 | 0 | 0 | 0 | $\cdots$ |
| $d=1$ | 0 | 1 | 1 | 1 | 1 | $\cdots$ |
| $d=2$ | 1 | 2 | 11 | 90 | – | |
| $d=3$ | 0 | 7 | – | – | – | |
| $d=4$ | 1 | 49 | – | – | – | |
| $d=5$ | 0 | – | – | – | – | |
| $\vdots$ | $\vdots$ | | | | | |

... coming from an ADE correspondence.

# ...to new cases

Table: Number of families of perfect ideals $I$ such that $S/I$ has Betti numbers $(1, 3+d, 2+d+t, t)$

| | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ | $\cdots$ |
|---|---|---|---|---|---|---|
| $d=0$ | 1 | 0 | 0 | 0 | 0 | $\cdots$ |
| $d=1$ | 0 | 1 | 1 | 1 | 1 | $\cdots$ |
| $d=2$ | 1 | 2 | 11 | 90 | – | |
| $d=3$ | 0 | 7 | – | – | – | |
| $d=4$ | 1 | 49 | – | – | – | |
| $d=5$ | 0 | – | – | – | – | |
| $\vdots$ | $\vdots$ | | | | | |

... coming from an ADE correspondence.

## ...to new cases

Table: Number of families of perfect ideals $I$ such that $S/I$ has Betti numbers $(1, 3+d, 2+d+t, t)$

| | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ | $\cdots$ |
|---|---|---|---|---|---|---|
| $d=0$ | 1 | 0 | 0 | 0 | 0 | $\cdots$ |
| $d=1$ | 0 | 1 | 1 | 1 | 1 | $\cdots$ |
| $d=2$ | 1 | 2 | 11 | 90 | – | |
| $d=3$ | 0 | 7 | – | – | – | |
| $d=4$ | 1 | 49 | – | – | – | |
| $d=5$ | 0 | – | – | – | – | |
| $\vdots$ | $\vdots$ | | | | | |

... coming from an ADE correspondence.