# AN ADE CORRESPONDENCE FOR GRADE THREE PERFECT IDEALS

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Abstract. Using the theory of "higher structure maps" from generic rings for free resolutions of length three, we give a classification of grade 3 perfect ideals with small type and deviation in local rings of equicharacteristic zero, extending the Buchsbaum-Eisenbud structure theorem on Gorenstein ideals and realizing it as the type D case of an ADE correspondence. We also deduce restrictions on Betti tables in the graded setting for such ideals.

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TODO: use $\mathbb{C}$ as the ground field and adjust/refer to §2.2.6 as appropriate. Add references/make corrections that Jerzy pointed out.

## 1. Introduction

We say an ideal $I$ in a local Noetherian $\mathbb{C}$-algebra $(R, \mathfrak{m})$ is *perfect* if $\operatorname{grade} I = \operatorname{pdim}_R R/I$, where $\operatorname{grade} I := \operatorname{depth}(I, R)$. In the event that $R$ is a regular local ring, $I$ being perfect is equivalent to $R/I$ being a Cohen-Macaulay $R$-module by the Auslander-Buchsbaum formula. It is well-known that all perfect ideals of grade two are determinantal. More precisely one has the following corollary of Hilbert-Burch:

**Theorem.** *Let $S$ be the polynomial ring on the variables $x_{i,j}$ ($1 \le i \le n$, $1 \le j \le n-1$), localized at the ideal of variables. Consider the complex*

$$\mathbb{G}\colon 0 \to S^{n-1} \xrightarrow{d_2} S^n \xrightarrow{d_1} S$$

*where $d_2 = [x_{i,j}]$ is the generic matrix in the variables $x_{i,j}$ and $(d_1)_{1,k}$ is $(-1)^k$ times the $k$-th $(n-1) \times (n-1)$ minor of $d_2$. This complex is acyclic.*

*Letting $J$ denote the image of $d_1$, if $I \subset R$ is a perfect ideal of grade two minimally generated by $n$ elements, then there exists a local homomorphism $\varphi\colon S \to R$ such that $\varphi(J)R = I$, or equivalently that $\mathbb{G} \otimes R$ resolves $R/I$.*

For perfect ideals of grade three, theorems of this type are only known in a few special cases. The Betti numbers $(1, b_1, b_2, b_3)$ of such an ideal are determined by the type $r(R/I) \geq 1$ and the deviation $d(I) \geq 0$: the former is the minimal number of generators for the canonical module of $R/I$, thus equal to $b_3$, and the latter is by definition $b_1 - 3$. Then one has $b_2 = b_1 + b_3 - 1 = r(R/I) + d(I) + 2$.

The case $r(R/I) = 1$ gives Gorenstein ideals, and these are characterized by the well-known structure theorem of Buchsbaum and Eisenbud: $I$ is generated by the $(n-1)\times(n-1)$ Pfaffians of an $n \times n$ skew matrix [3] which appears as the differential $d_2$ in a resolution of $R/I$. From this, an analogous result for almost complete intersections (i.e. $d(I) = 1$) can be deduced from linkage. This was also carried out in [3], and then made more explicit in [1].

The natural question to pose is whether all perfect ideals of grade 3 with a fixed type and deviation can be realized as specializations of some generic example, as they do in these two cases. The smallest new case to consider would be when $r(R/I) = d(I) = 2$, so $R/I$ has Betti numbers $(1, 5, 6, 2)$. One observation here is that the multiplication

$$\operatorname{Tor}_1(R/I, k) \otimes \operatorname{Tor}_1(R/I, k) \to \operatorname{Tor}_2(R/I, k)$$

may or may not be zero, where $k = R/\mathfrak{m}$. If $(R', \mathfrak{m}')$ is another local ring and $\varphi\colon R \to R'$ is a local homomorphism such that $I' = \varphi(I)R'$ is a perfect ideal of grade three in $R'$, then $\varphi$ induces an inclusion of residue fields $k \to k' \coloneqq R'/\mathfrak{m}'$ through which $\operatorname{Tor}_*(R'/I', k') = \operatorname{Tor}_*(R/I, k) \otimes k'$. Consequently the property of this multiplication being (non)zero is preserved under local specialization, meaning that there cannot be a single generic example for perfect ideals with Betti numbers $(1, 5, 6, 2)$.

As we will see, the next best thing is true: there is a generic example for each of the two cases. If the multiplication is nonzero, the ideal is directly linked to one with Betti numbers $(1, 4, 5, 2)$ and the generic example is given in [1]. An ideal $J(t)$ parametrizing a family with zero multiplication on $\operatorname{Tor}_*(R'/I', k')$ was studied in [6] and revisited in [11] in a characteristic-free manner. In fact, the ideal $J(t)$ is the generic example for that case and we will establish this as a corollary of our main classification result Theorem 4.4, which explains these two families in terms of the representation theory of $E_6$.

To explain the ADE correspondence, we revisit the Gorenstein case. Let $n = d(I) + 3$ so that the Betti numbers are $(1, n, n, 1)$. If $\mathbb{F}$ is an arbitrary minimal free resolution of $R/I$, then of course the differential $d_2\colon F_2 \to F_1$ need not be a skew matrix. The skew matrix only appears after one identifies $F_2 \cong F_1^*$ using a choice of multiplication $F_1 \otimes F_2 \to F_3$ on the free resolution.

We can rephrase this as follows. The differential $d_2\colon F_2 \to F_1$ and the multiplication $F_1 \otimes F_2 \to F_3$ can be put side by side as an $n \times 2n$ matrix

$$\begin{array}{cc} & \begin{array}{cc} F_1^* & F_1 \end{array} \\ F_2^* & \begin{bmatrix} d_2^* & w_1^{(2)} \end{bmatrix} \end{array}$$

where $w_1^{(2)}$ is the multiplication viewed as an isomorphism $F_1 \cong F_3^* \otimes F_1 \to F_2$. This matrix determines a map from $\operatorname{Spec} R$ to the orthogonal Grassmannian $OG(n, 2n)$ of isotropic $n$-planes inside of $F_1^* \oplus F_1$ with the quadratic form $Q$ given by the evident pairing. To see this, there is an affine patch $N \subset OG(n, 2n)$ consisting of those isotropic $n$-planes represented by an $n \times 2n$ matrix where the last $n \times n$ minor is non-vanishing. On such a matrix, after performing row operations so that the right $n \times n$ block is the identity matrix, the condition that $Q = 0$ is equivalent to the left block being skew. Moreover, on the affine patch $N$, the $(n-1)\times(n-1)$ pfaffians of the left skew matrix give Plücker coordinates cutting out a particular Schubert variety $X \subset OG(n, 2n)$ of codimension 3. Thus the Buchsbaum-Eisenbud structure theorem yields a map $\operatorname{Spec} R \to N$, through which the local defining equations of $X$ at the "origin" in $N$ (representing the isotropic $n$-plane $F_1 \subset F_1^* \oplus F_1$) pull back to the generators of the ideal $I \subset R$. For a thorough explanation of this perspective, we refer to [8].

We will show that this formulation of the Buchsbaum-Eisenbud structure theorem realizes it as the type $D$ case of an ADE correspondence. In general, to deviation $d(I)$ and type $r(R/I)$, we associate the $T$-shaped graph with arms of length 1, $d(I)$, and $r(R/I)$. This graph $T$ is the Dynkin diagram

- $A_n$ if $d(I) = 0$ and $r(R/I) = n-2$,
- $D_n$ if $d(I) = 1$ and $r(R/I) = n-3$ or vice versa,
- $E_6$ if $d(I) = r(R/I) = 2$,
- $E_7$ if $d(I) = 2$ and $r(R/I) = 3$ or vice versa,
- $E_8$ if $d(I) = 2$ and $r(R/I) = 4$ or vice versa.

Let $G$ be the associated group, and let $x_1 \in T$ be the node on the arm of length 1. We will show that generic examples of perfect ideals with the given type and deviation come from a certain codimension 3 Schubert variety inside of the homogeneous space $G/P_{x_1}$ where $P_{x_1}$ is the maximal parabolic for the node $x_1$.

The type $A$ case of the correspondence is uninteresting, as it occurs only when $d(I) = 0$. This necessarily means that $I$ is generated by a regular sequence—in particular, $r(R/I) = 1$. The homogeneous space is $G/P_{x_1} = SL_3/P_1 = \mathbb{P}^3$ in this case, and the Schubert variety $X$ is a point, which is indeed a complete intersection.

We mentioned above that there are two different families of perfect ideals with $d(I) = r(R/I) = 2$. This happens because the map $\operatorname{Spec} k \to \operatorname{Spec} R \to G/P_{x_1} = E_6/P_2$ lands in one of two strata of $X \subset E_6/P_2$, depending on whether the aforementioned multiplication on the Tor algebra is zero. We will describe these strata in §4 and revisit this in Example 4.8.

To achieve our goals, the main tool we will need is an appropriate generalization of the $n \times 2n$ block matrix leveraged in the Gorenstein case, and this is provided by the theory of "higher structure maps" originating from Weyman's generic rings for resolutions of length three. We now briefly explain this; more details will be given in §2. Given a complex

$$\mathbb{F}\colon 0 \to F_m \xrightarrow{d_m} \cdots \to F_1 \xrightarrow{d_1} F_0,$$

where $F_i = R^{f_i}$, we refer to the sequence $(f_0, f_1, \ldots, f_m)$ as the *format* of $\mathbb{F}$. In the event that $\mathbb{F}$ is a minimal resolution over a local ring $R$, the $f_i$ are the ordinary Betti numbers of the module $H_0(\mathbb{F})$, but we will benefit from working in greater generality. For each fixed format $(f_0, f_1, f_2, f_3)$ of length three, Weyman constructed in [19] and [18] a resolution $\mathbb{F}^{\text{gen}}$ over a ring $\widehat{R}_{\text{gen}}$ with the property that it specializes to any free resolution of the given format.

This ring is a finitely generated $\mathbb{C}$-algebra if and only if the format $(f_0, f_1, f_2, f_3)$ is one listed in Table 1. These are called *Dynkin formats*, because the structure of $\widehat{R}_{\text{gen}}$ is closely tied to a Kac-Moody Lie algebra $\mathfrak{g}$ that is finite-dimensional (i.e. of Dynkin type) exactly in these cases. Henceforth we will always assume this to be the case.

We will explain the Lie algebra $\mathfrak{g}$ and its relation to $\widehat{R}_{\text{gen}}$ more precisely in §2. For now we comment that there are three representations inside of $\widehat{R}_{\text{gen}}$ of particular interest: namely those generated by the entries of the differentials $d_i$ of $\mathbb{F}^{\text{gen}}$. We call these the *critical representations*. They have a graded decomposition in which each graded component is a representation of $\prod GL(F_i)$ where $F_i = \mathbb{C}^{f_i}$.

TABLE 1. Length three formats with Noetherian $\widehat{R}_{\text{gen}}$

| Type $D_n$ | Type $E_6$ | Type $E_7$ | Type $E_8$ | |
|---|---|---|---|---|
| $(1,n,n,1)$ | $(1,5,6,2)$ | $(1,6,7,2)$ | $(1,7,8,2)$ | Format I (dual to VI) |
| $(1,4,n,n-3)$ | | $(1,5,7,3)$ | $(1,5,8,4)$ | Format II (linked to I) |
| $(n-3,n,4,1)$ | $(2,6,5,1)$ | $(3,7,5,1)$ | $(4,8,5,1)$ | Format III (dual to II) |
| | $(2,5,5,2)$ | $(3,6,5,2)$ | $(4,7,5,2)$ | Format IV (linked to III) |
| | | $(2,5,6,3)$ | $(2,5,7,4)$ | Format V (dual to IV) |
| | | $(2,7,6,1)$ | $(2,8,7,1)$ | Format VI (linked to V) |

We have displayed pieces of them below.

$$W(d_3) = F_2^* \otimes [F_3 \oplus \bigwedge^{f_0+1} F_1 \oplus \cdots]$$
$$W(d_2) = F_2 \otimes [F_1^* \oplus F_3^* \otimes \bigwedge^{f_0} F_1 \oplus \cdots]$$
$$W(d_1) = F_0^* \otimes [F_1 \oplus F_3^* \otimes \bigwedge^{f_0+2} F_1 \oplus \cdots]$$

Given a homomorphism $w\colon \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}$, restriction to $W(d_i)$ yields maps

$$w^{(3)}\colon R \otimes [F_3 \oplus \bigwedge^{f_0+1} F_1 \oplus \cdots] \to R \otimes F_2$$
$$w^{(2)}\colon R \otimes [F_1^* \oplus F_3^* \otimes \bigwedge^{f_0} F_1 \oplus \cdots] \to R \otimes F_2^*$$
$$w^{(1)}\colon R \otimes [F_1 \oplus F_3^* \otimes \bigwedge^{f_0+2} F_1 \oplus \cdots] \to R \otimes F_0$$

By abuse of notation we will sometimes write $F_i$ to mean $F_i \otimes R$ when the meaning can be inferred from context. We write $w_j^{(i)}$ for the $j$-th component of the map $w^{(i)}$, with indexing starting at $j = 0$. For instance $w_0^{(3)}\colon F_3 \to F_2$ is just the differential $d_3$ of $\mathbb{F}$. Likewise $w_0^{(2)} = d_2^*$ and $w_0^{(1)} = d_1$. If $\mathbb{F}$ resolves $R/I$ for some ideal $I$ of grade at least[1] 2, then the maps $w_1^{(i)}$ give a choice of multiplicative structure on $\mathbb{F}$ lifting that on $\mathrm{Tor}_*(R/I,k)$. In general we will refer to the maps $w^{(i)}$ and their components $w_j^{(i)}$ as "(higher) structure maps" for the resolution $\mathbb{F}$.

The main technical result we will establish in §3 is that the surjectivity of the maps $w^{(i)}$ is guaranteed if the Betti numbers of $R/I$ are Dynkin. This should be viewed as the substitute for the perfect pairing $F_1 \otimes F_2 \to F_3$ used in the Gorenstein case to identify $F_1 \cong F_2^*$, which gave surjectivity of the $n \times 2n$ matrix.

**Theorem 3.1.** Suppose that $\mathbb{F}$ is a resolution of Dynkin format over a $\mathbb{Q}$-algebra $R$ such that its dual $\mathbb{F}^*$ is also acyclic. Then if $w\colon \widehat{R}_{\text{gen}} \to R$ specializes $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}$, the maps $w^{(i)}$ are surjective.

**Remark 1.1.** Also of interest is the representation $W(a_2)$ generated by the entries of the Buchsbaum-Eisenbud multiplier $a_2$. The product of this representation with $a_1$ appears in the subrepresentation $\bigwedge^{f_0} F_0^* \otimes \bigwedge^{f_0}[F_1 \oplus \cdots]$ of $S_{f_0} W(d_1)$, which is to say the $f_0 \times f_0$ minors of $w^{(1)}$. We will show that

[1]To be precise, it gives a choice of multiplicative structure on $0 \to F_3 \to F_2 \to F_1 \xrightarrow{a_2} R$ where $a_2$ comes from the First Structure Theorem of [4]. If $I$ has grade at least 2, then it is equal to the image of $a_2$.

the restriction of $w$ to $W(a_2)$ is surjective as well, but the importance of considering $W(a_2)$ is only apparent when dealing with module formats with $f_0 > 1$.

In §2, we provide background on Lie algebras, $\widehat{R}_{\text{gen}}$, and Schubert varieties. Then we will prove Theorem 3.1 in §3, and deduce some restrictions on graded Betti tables as a corollary. If $\mathbb{F}$ resolves $R/I$ for a grade three perfect ideal $I$ in a local ring $(R, \mathfrak{m})$, the surjectivity of $w^{(1)}$ in Theorem 3.1 is equivalent to the map being nonzero mod $\mathfrak{m}$. Given this, we can ask for the first component of this structure map that is nonzero mod $\mathfrak{m}$. After posing this question more precisely, we will show in §4 that it has a well-defined answer for each ideal $I$, which may be used to classify the ideal. As an example in a very simple case, if $w_1^{(1)}\colon F_3^* \otimes \bigwedge^3 F_1 \to R$ is nonzero mod $\mathfrak{m}$, then $I$ is a complete intersection. From the geometric perspective, this first nonzero component determines the stratum of the Schubert variety $X \subset G/P_{x_1}$ in which the map $\operatorname{Spec} k \to \operatorname{Spec} R \to G/P_{x_1}$ lands, and using this we show how the local defining equations of $X \subset G/P_{x_1}$ yield generic perfect ideals. Theorem 3.1 fails without the Dynkin hypothesis: the ideal $I = (x, y, z)^2 \subset R = \mathbb{C}[x, y, z]_{(x,y,z)}$ is perfect of grade three, with $r(R/I) = 3$ and $d(I) = 3$ ($T = \widetilde{E}_7$ is not Dynkin), and one can show that all entries of $w^{(i)}$ lie in $\mathfrak{m} = (x, y, z)$ so the maps cannot be surjective. This failure of Theorem 3.1 is related to the theory of linkage, and this will be discussed further in §5, together with other directions for future study.

## 2. Background

In this section we gather the necessary background on Lie algebras in §2.1, the generic ring $\widehat{R}_{\text{gen}}$ in §2.2, and Schubert varieties in §2.3. We will need §2.1 and §2.2 for §3. The background given in §2.3 will be needed for §4. To streamline the exposition, we will work in the category of $\mathbb{C}$-algebras, but as we discuss in §2.2.6, our main results hold over $\mathbb{Q}$.

### 2.1. **Lie algebras.**

2.1.1. *Construction.* Fix integers $p, q, r \geq 1$, and let $T$ denote the graph

$$\begin{array}{ccccccccc}
x_{p-1} & — & \cdots & — & x_1 & — & u & — & y_1 & — & \cdots & — & y_{q-1} \\
 & & & & & & | & & & & & & \\
 & & & & & & z_1 & & & & & & \\
 & & & & & & | & & & & & & \\
 & & & & & & \vdots & & & & & & \\
 & & & & & & | & & & & & & \\
 & & & & & & z_{r-1} & & & & & &
\end{array}$$

Let $n = p + q + r - 2$ be the number of nodes. From the above graph, we construct an $n \times n$ matrix $A$, called the *Cartan matrix*, whose rows and columns are indexed by the nodes of $T$:

$$A = (a_{i,j})_{i,j \in T}, \quad a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i, j \in T \text{ are adjacent}, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper we will only consider the situation when $T$ is a Dynkin diagram. In terms of the parameters $p$, $q$, and $r$, this is equivalent to the inequality $1/p + 1/q + 1/r > 1$. We next describe how to construct the associated Lie algebra $\mathfrak{g}$.

Let $\mathfrak{h} = \mathbb{C}^n$, and take $\Pi = \{\alpha_i\}_{i\in T}$ in $\mathfrak{h}^*$ to be the coordinate functions. These are the *simple roots*. Let $\Pi^\vee = \{\alpha_i^\vee\}_{i\in T}$ be elements of $\mathfrak{h}$ such that

$$\langle \alpha_i^\vee, \alpha_j \rangle = a_{i,j}.$$

These are the *simple coroots*. The Lie algebra $\mathfrak{g} := \mathfrak{g}(T)$ is generated by elements $e_i, f_i$ for $i \in T$, subject to the defining relations

$$\begin{gathered}
[e_i, f_j] = \delta_{i,j}\alpha_i^\vee, \\
[h, e_i] = \langle h, \alpha_i\rangle e_i, [h, f_i] = -\langle h, \alpha_i\rangle f_i \text{ for } h \in \mathfrak{h}, \\
[h, h'] = 0 \text{ for } h, h' \in \mathfrak{h}, \\
\mathrm{ad}(e_i)^{1-a_{i,j}}(e_j) = \mathrm{ad}(f_i)^{1-a_{i,j}}(f_j) \text{ for } i \neq j.
\end{gathered}$$

Under the adjoint action of $\mathfrak{h}$, the Lie algebra $\mathfrak{g}$ decomposes into eigenspaces as $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$, where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

This is the *root space decomposition* of $\mathfrak{g}$.

2.1.2. *Gradings on* $\mathfrak{g}$. Let $Q \subset \mathfrak{h}^*$ be the root lattice $\bigoplus_{i\in T} \mathbb{Z}\alpha_i$. If $\mathfrak{g}_\alpha \neq 0$, then necessarily $\alpha \in Q$. If such an $\alpha$ is nonzero, we say it is a *root*. Hence the Lie algebra $\mathfrak{g}$ is $Q$-graded. By singling out a vertex $t \in T$, this $Q$-grading can be coarsened to a $\mathbb{Z}$-grading by considering only the coefficient of $\alpha_t$. We refer to this as the $t$-grading. The sum of all $t$-gradings for $t \in T$ is called the *principal gradation* on $\mathfrak{g}$. The degree zero part in the principal gradation is the Cartan subalgebra $\mathfrak{h}$.

Using these notions, we define a few important subalgebras of $\mathfrak{g}$:

- Let $\mathfrak{n}^+, \mathfrak{n}^-$ denote the positive and negative parts of $\mathfrak{g}$ in the principal gradation.
- Let $\mathfrak{b}^+, \mathfrak{b}^-$ denote the nonnegative and nonpositive parts of $\mathfrak{g}$ in the principal gradation. That is, $\mathfrak{b}^+ = \mathfrak{n}^+ + \mathfrak{h}$ and $\mathfrak{b}^- = \mathfrak{n}^- + \mathfrak{h}$.
- For $t \in T$, let $\mathfrak{n}_t^+, \mathfrak{n}_t^-$ denote the positive and negative parts of $\mathfrak{g}$ in the $t$-grading.
- For $t \in T$, let $\mathfrak{p}_t^+, \mathfrak{p}_t^-$ denote the nonnegative and nonpositive parts of $\mathfrak{g}$ in the $t$-grading.

Write $h_i \in \mathfrak{h}$ for the basis dual to the simple roots $\alpha_i \in \mathfrak{h}^*$. The degree zero part of $\mathfrak{g}$ in the $t$-grading is

$$\mathfrak{g}^{(t)} \times \mathbb{C}h_t$$

where $\mathfrak{g}^{(t)}$ is the subalgebra generated by $\{e_i, f_i\}_{i\neq t}$ and $\mathbb{C}h_t$ is the one-dimensional abelian Lie algebra spanned by $h_t$. The decomposition of $\mathfrak{g}$ into $t$-graded components is just its decomposition into eigenspaces for the adjoint action of $h_t$:

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \ker(\mathrm{ad}(h_t) - j).$$

We will be primarily interested in the $t$-grading when $t \in \{x_1, z_1\}$.

- The diagram $T - \{x_1\}$ consists of a diagram $A_{p-2}$ with vertices $x_2, \ldots, x_{p-1}$ and $A_{q+r-1}$ with vertices $y_{q-1}, \ldots, y_1, u, z_1, \ldots, z_{r-1}$. Hence writing $F_0 = \mathbb{C}^{p-1}$ and $F_2 = \mathbb{C}^{q+r}$, we identify $\mathfrak{g}^{(x_1)} = \mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2)$.
- The diagram $T - \{z_1\}$ consists of a diagram $A_{p+q-1}$ with vertices $y_{q-1}, \ldots, y_1, u, x_1, \ldots, x_{p-1}$ and $A_{r-2}$ with vertices $z_2, \ldots, z_{r-1}$. Hence writing $F_1 = \mathbb{C}^{p+q}$ and $F_3 = \mathbb{C}^{r-1}$, we identify $\mathfrak{g}^{(z_1)} = \mathfrak{sl}(F_1) \times \mathfrak{sl}(F_3)$.

2.1.3. *Representations.* Let $V$ be a representation of $\mathfrak{g}$. For $\lambda \in \mathfrak{h}$, define the *$\lambda$-weight space of $V$* to be

$$V_\lambda = \{v \in V : hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\lambda \neq 0$, then we say $\lambda$ is a *weight* of $V$. A nonzero vector $v \in V_\lambda$ is a *highest weight vector* if $\mathfrak{n}^+ v = 0$. If such a $v$ generates $V$ as a $\mathfrak{g}$-module, then we say $V$ is a *highest weight module* with highest weight $\lambda$.

Let $\mathcal{U}$ denote the universal enveloping algebra functor. Representations of $\mathfrak{g}$ are equivalent to modules over $\mathcal{U}(\mathfrak{g})$. Given $\lambda \in \mathfrak{h}^*$, the *Verma module* $M(\lambda)$ is defined to be

$$M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \mathbb{C}_\lambda.$$

Here $\mathbb{C}_\lambda$ is the $\mathfrak{b}^+$-module where $\mathfrak{h}$ acts by $\lambda$ and $\mathfrak{n}^+$ acts trivially. All the weights of $M(\lambda)$ are in $\lambda + Q$. If $v \in V_\lambda$ is a highest weight vector, then there is a map $M(\lambda) \to V$ sending $1 \mapsto v$. If $V$ is a highest weight module then this map is surjective.

Every Verma module $M(\lambda)$ has a unique maximal proper submodule $J(\lambda)$, namely the sum of all submodules which do not contain $v$. It follows that $V(\lambda) = M(\lambda)/J(\lambda)$ is an irreducible highest weight module with heighest weight $\lambda$, and any such module is isomorphic to $V(\lambda)$.

Let $\omega_i \in \mathfrak{h}^*$ be the basis dual to $\alpha_i^\vee \in \mathfrak{h}$. Explicitly, $\omega_i$ is the linear combination of $\alpha_i$ given by the $i$-th column of $A^{-1}$. These are the *fundamental weights*, and the representations $V(\omega_i)$ are called *fundamental representations*. Their nonnegative integral span is the collection of *dominant weights*, and the representation $V(\lambda)$ is finite-dimensional when $\lambda$ is dominant.

One can alternatively work with lowest weights instead of highest weights, interchanging the roles of positive and negative parts of the Lie algebra in all of the preceding. We will write $V(\lambda)^\vee$ for the irreducible representation with lowest weight $-\lambda$. If $V(\lambda)$ is finite-dimensional, then $V(\lambda)^* = V(\lambda)^\vee$. In general, $(-)^\vee$ represents the "restricted" dual.

**Remark 2.1.** It is possible to define a linear algebraic group $G$ with associated Lie algebra $\mathfrak{g}$ along the lines of §2.1.1; see also the discussion at the beginning of §2.3. We will not need it, so we omit doing so. But from §2.1.2, we certainly have the actions of the groups $SL(F_i)$ on $\mathfrak{g}$ and its representations $V(\lambda)$ whenever $\lambda$ is dominant and integral, and we will make use of this.

2.1.4. *Weight grading on representations.* The decomposition of $V(\lambda)$ into weight spaces gives an $\mathfrak{h}^*$-grading on $V(\lambda)$. Moreover, all the weights of $V(\lambda)$ are in the translate $\lambda + \bigoplus_{i\in T} \mathbb{Z}\alpha_i$ of the root lattice.

In §2.1.2 it was described how singling out a vertex $t \in T$ allows us to impose a $\mathbb{Z}$-grading on $\mathfrak{g}$ by considering only the coefficient of $\alpha_t$ in the $\mathfrak{h}^*$-grading. This works for representations $V(\lambda)$ as well: if $v \in V(\lambda)$ is a highest weight vector then $h_t v = \langle h_t, \lambda\rangle v$ and the eigenvalues for the action of $h_t$ on $V(\lambda)$ are $\langle h_t, \lambda\rangle, \langle h_t, \lambda\rangle - 1, \ldots$, terminating iff $V(\lambda)$ is finite-dimensional. The eigenspaces give the $t$-graded components. Each one is a representation of the subalgebra $\mathfrak{g}^{(t)} \times \mathbb{C}h_t \subset \mathfrak{g}$. In particular, $v$ is a highest weight vector for the top graded component, thus this component is the representation of $\mathfrak{g}^{(t)}$ with highest weight $\sum_{i\neq t} c_i\omega_i$ if $\lambda = \sum_{i\in T} c_i \omega_i$.

2.1.5. *Exponential action and Baker-Campbell-Hausdorff.* Let $\bigoplus_{i>0} \mathbb{L}_i$ be a strictly positively graded Lie algebra. Its bracket naturally extends to one on $\mathbb{L} = \prod_{i>0} \mathbb{L}_i$. (For us, this Lie algebra will generally be $\mathfrak{n}_t^\pm$ for some $t \in T$, and since we are only considering Dynkin $T$, there are only finitely many components and thus the direct sum and product are the same.) Suppose $R$ is an $R_0$-algebra on which $\mathbb{L}$ acts by locally nilpotent $R_0$-linear derivations. Here "locally nilpotent" means that for any

Lie algebra element $X$ and ring element $f \in R$, we have $X^N f = 0$ for $N \gg 0$. Then the exponential

$$\exp X = \mathrm{Id} + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots$$

defines an $R_0$-algebra automorphism of $R$.

Moreover, given Lie algebra elements $X, Y \in \mathbb{L}$, the Baker-Campbell-Hausdorff formula gives a well-defined element $Z \in \mathbb{L}$ such that $\exp Z = (\exp X)(\exp Y)$:

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots.$$

We will not need the explicit expression for $Z$, only that such an expression exists in terms of iterated commutators.

## 2.2. **The generic ring $\widehat{R}_{\mathrm{gen}}$.**

2.2.1. *Generic free resolutions.* Let $\mathbb{F}$ be a complex of free modules

$$\mathbb{F}\colon 0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to F_0$$

over some ring $R$. The *format* of $F$ is the sequence of numbers $(f_0, f_1, \ldots, f_c)$ where $f_i = \operatorname{rank} F_i$.

Let $S$ be a ring and $\mathbb{G}$ a resolution over $S$ with format $(f_0, \ldots, f_c)$. We say that $(S, \mathbb{G})$ is *generic* for the given format if, given any resolution $\mathbb{F}$ of the same format over some ring $R$, there exists a map $\varphi\colon S \to R$ specializing $\mathbb{G}$ to $\mathbb{F}$. Note that we do not require the map $\varphi$ to be unique: Bruns showed that this is a mandatory concession for $c \geq 3$ [2]. We will only be concerned with $c = 3$ in this paper and we henceforth restrict to this case.

2.2.2. *The first structure theorem of Buchsbaum and Eisenbud.* The first structure theorem of [4] is essential to the construction of Weyman's generic ring. We recall it here with two small adjustments: we state it only for $c = 3$ and we avoid identifying top exterior powers with the base ring in the interest of doing things $GL(F_i)$-equivariantly.

**Theorem 2.2.** *Let $0 \to F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$ be a complex of free modules, acyclic in grade 1, of format $(f_0, f_1, f_2, f_3)$ over $R$. Then there are uniquely determined maps $a_3, a_2, a_1$, constructed as follows:*

- *$a_3$ is the top exterior power*
$$a_3\colon \bigwedge^{f_3} F_3 \to \bigwedge^{f_3} F_2.$$
- *$a_2$ is the unique map making the following diagram commute, where $r_2 = f_2 - f_3 = f_1 - f_0$ is the rank of the differential $d_2$:*
$$\begin{tikzcd} \bigwedge^{r_2} F_2 \arrow[rr] \arrow[dr, "-\wedge a_3"'] & & \bigwedge^{r_2} F_1 \\ & \bigwedge^{f_3} F_3^* \otimes \bigwedge^{f_2} F_2 \arrow[ur, dashed, "a_2"'] & \end{tikzcd}$$
- *$a_1$ is the unique map making the following diagram commute:*
$$\begin{tikzcd} \bigwedge^{f_0} F_1 \arrow[rr] \arrow[dr, "-\wedge a_2"'] & & \bigwedge^{f_0} F_0 \\ & \bigwedge^{f_3} F_3 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_1} F_1 \arrow[ur, dashed, "a_1"'] & \end{tikzcd}$$

Note that $a_1$ is just a scalar. If grade $I_{f_0}(d_1) \geq 2$, then $a_1$ is an isomorphism, and we will use this in §3.

2.2.3. *Construction of* $\widehat{R}_{\text{gen}}$. Fix a length 3 format $(f_0, f_1, f_2, f_3)$. The generic pair $(\widehat{R}_{\text{gen}}, \mathbb{F}^{\text{gen}})$ was constructed by Weyman in [19], and the acyclicity of $\mathbb{F}^{\text{gen}}$ was proven in [18]. We will only summarize some details of the construction here.

Theorem 2.2 can be used to construct a free complex $\mathbb{F}^a$ of the given format over a ring $R_a$, called the Buchsbaum-Eisenbud multiplier ring, such that $\mathbb{F}^a$ is acyclic in grade 1 and is universal with respect to this property. In particular, if $\mathbb{F}$ is any resolution of the same format over some ring $R$, then there is a unique map $R_a \to R$ specializing $\mathbb{F}^a$ to $\mathbb{F}$.

However, $\mathbb{F}^a$ is not acyclic: letting $d_i$ denote the differentials of $\mathbb{F}^a$ and $r_i$ their ranks, we have

$$\operatorname{grade} I_{r_3}(d_3) = 2, \quad \operatorname{grade} I_{r_2}(d_2) = 2, \quad \operatorname{grade} I_{r_1}(d_1) = 1.$$

We recall the Buchsbaum-Eisenbud acyclicity criterion from [5]: a finite free complex is acyclic exactly when grade $I_{r_i}(d_i) \geq i$ and $f_i = r_{i+1} + r_i$ for all $i$. The latter condition on rank is satisfied, but the grade of $I_{r_3}(d_3)$ is too low.

This leads to the basic idea underlying the construction of Weyman's generic pair $(\widehat{R}_{\text{gen}}, \mathbb{F}^{\text{gen}})$: kill $H^2$ in the Koszul complex on $\bigwedge^{r_3}(d_3)$ to increase the grade of $I_{r_3}(d_3)$ from 2 to 3. It is this process that introduces the non-uniqueness of the map $\varphi$ alluded to in §2.2.1. Explicitly, writing $\mathcal{K} = \bigwedge^{f_3} F_3^* \otimes \bigwedge^{f_3} F_2$, this is an inductive procedure illustrated in the diagram below.

$$\begin{tikzcd} 0 \arrow[r] & \bigwedge^0 \mathcal{K} \arrow[r] & \bigwedge^1 \mathcal{K} \arrow[r] & \bigwedge^2 \mathcal{K} \arrow[r] & \bigwedge^3 \mathcal{K} \\ & & \mathbb{L}_i^* \arrow[u, "p_i"] \arrow[r] & (\bigwedge^2 \mathbb{L}^*)_i \arrow[u, "q_i"] & \end{tikzcd} \tag{2.1}$$

Here $\mathbb{L}$ is a graded Lie algebra, called the *defect Lie algebra*, and the lower horizontal map is dual to the bracket in $\mathbb{L}$. The map $q_1$ is defined using the Second Structure Theorem of [4], and $q_m$ for $m \geq 2$ is defined using $p_i$ for $i < m$. For positive integers $m$, define $R_m$ to be the ring obtained from $R_a$ by adjoining variables for the coordinates of $p_1, \ldots, p_m$, quotienting by all relations they would satisfy on a split exact complex (see for instance [19, Lemma 2.4]), and taking the ideal transform with respect to $I_{r_2}(d_2)I_{r_3}(d_3)$. The ring $\widehat{R}_{\text{gen}}$ is defined to be the limit of the rings $R_m$, and $\mathbb{F}^{\text{gen}} := \mathbb{F}^a \otimes \widehat{R}_{\text{gen}}$.

2.2.4. *Exponential action of the defect Lie algebra.* Given a free resolution $\mathbb{F}$ over some ring $R$, to determine a map $\widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}$, it is sufficient to specify the images of the maps $p_i$ in $R$. Having chosen $p_i$ for $i < m$, the diagram (2.1) shows that the non-uniqueness of $p_m$ lifting $q_m$ is exactly $\operatorname{Hom}(\mathbb{L}_m^*, \bigwedge^0 \mathcal{K}) = \mathbb{L}_m \otimes R$. In [19], the action of $\mathbb{L}$ on $\widehat{R}_{\text{gen}}$ by derivations is described, and it was observed in [12, Theorem 3.1] that different choices of maps $\widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to the same resolution $\mathbb{F}$ are related by the exponential action of a unique element of $\mathbb{L} \otimes R$ on $R \otimes \widehat{R}_{\text{gen}}$. We restate the result here:

**Theorem 2.3.** *Let $\mathbb{F}$ be a resolution of length three over $R$ and let $\widehat{R}_{\text{gen}}$ be the generic ring for the associated format. Fix a $\mathbb{C}$-algebra homomorphism $w\colon \widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}$. Then $w$ determines a bijection*

$$\mathbb{L} \otimes R \simeq \{\mathbb{C}\text{-}\textit{algebra homomorphisms } w'\colon \widehat{R}_{\text{gen}} \to R \textit{ specializing } \mathbb{F}^{\text{gen}} \textit{ to } \mathbb{F}\}.$$

*Note that a $\mathbb{C}$-algebra homomorphism $\widehat{R}_{\text{gen}} \to R$ can be viewed as an $R$-algebra homomorphism $R \otimes \widehat{R}_{\text{gen}} \to R$. The correspondence above identifies $X \in \mathbb{L} \otimes R$ with the map $w \exp X$ obtained by precomposing $w$ with the action of* $\exp X$ *on* $R \otimes \widehat{R}_{\text{gen}}$.

Thus the defect Lie algebra parametrizes the collection of maps $\widehat{R}_{\text{gen}} \to R$ specializing $\mathbb{F}^{\text{gen}}$ to $\mathbb{F}$ in a very precise sense.

2.2.5. *The critical representations and higher structure maps.* One of the main results of [18] is that the defect Lie algebra $\mathbb{L}$ is a subalgebra of the Lie algebra $\mathfrak{g}$ defined in §2.1 for the diagram $T$ with $p = f_0 + 1$, $q = f_1 - f_0 - 1$, and $r = f_3 + 1$. Specifically, $\mathbb{L}$ is the negative part[2] $\mathfrak{n}_{z_1}^-$ of $\mathfrak{g}$ in the grading induced by the vertex $z_1$ (c.f. §2.1.2).

The actions of $\mathfrak{sl}(F_i)$ and $\mathbb{L}$ on $\widehat{R}_{\text{gen}}$ can be combined and extended to an action of $\mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{g}$. (The reason we say $\mathfrak{sl}(F_i)$ here instead of $\mathfrak{gl}(F_i)$ is somewhat subtle; see Remark 2.4.) The decomposition of $\widehat{R}_{\text{gen}}$ into representations for this product is detailed in [18]. Of these representations, there are a few of particular interest, which we call the *critical representations*—they are the ones generated by the entries of the differentials $d_i$ and Buchsbaum-Eisenbud multipliers $a_i$ for $\mathbb{F}^{\text{gen}}$. We denote these representations by $W(d_i)$ and $W(a_i)$ respectively. Let $V_-(-\lambda)$ be the irreducible representation with lowest weight $-\lambda$ (c.f. §2.1) and let $M = \bigwedge^{f_3} F_3 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_1} F_1$. The aforementioned representations are

$$
\begin{aligned}
W(d_3) &= F_2^* \otimes V_-(-\omega_{z_{r-1}}) \\
&= F_2^* \otimes [F_3 \oplus M^* \otimes \bigwedge^{f_0+1} F_1 \oplus \cdots] \\
W(d_2) &= F_2 \otimes V_-(-\omega_{y_{q-1}}) \\
&= F_2 \otimes [F_1^* \oplus M^* \otimes F_3^* \otimes \bigwedge^{f_0} F_1 \oplus \cdots] \\
W(d_1) &= F_0^* \otimes V_-(-\omega_{x_{p-1}}) \\
&= F_0^* \otimes [F_1 \oplus M^* \otimes F_3^* \otimes \bigwedge^{f_0+2} F_1 \oplus \cdots] \\
W(a_3) &= \bigwedge^{f_3} F_2^* \otimes V_-(-\omega_{z_1}) \\
&= \bigwedge^{f_3} F_2^* \otimes [\bigwedge^{f_3} F_3 \oplus \cdots] \\
W(a_2) &= \bigwedge^{f_2} F_2 \otimes V_-(-\omega_{x_1}) \\
&= \bigwedge^{f_2} F_2 \otimes [\bigwedge^{r_2} F_1^* \otimes \bigwedge^{f_3} F_3^* \oplus \cdots] \\
W(a_1) &= \bigwedge^{f_0} F_0^* \otimes \bigwedge^{f_1} F_1 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_3} F_3
\end{aligned} \tag{2.2}
$$

**Remark 2.4.** One subtlety to point out is that while the decomposition of each $\mathfrak{g}$-representation into $\mathfrak{sl}(F_3) \times \mathfrak{sl}(F_1)$-representations follows from §2.1.2 using the vertex $z_1 \in T$, the actions of $\mathfrak{gl}(F_i)$ are *not* entirely visible from that of $\mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{g}$. This is clear for $W(a_1)$ as an example: it is the trivial representation for this product, but it is not the trivial representation for $\prod \mathfrak{gl}(F_i)$. To see

[2]To be more precise, it is the "completion" of the negative part of $\mathfrak{g}$ where one takes the product of the graded components instead of the direct sum as mentioned in §2.1.5, but this only makes a difference when $\mathfrak{g}$ is infinite-dimensional (i.e. $T$ is not a Dynkin diagram). We do not consider that case here.

the actions of $\prod \mathfrak{gl}(F_i)$ and $\mathfrak{g}$ together on $\widehat{R}_{\text{gen}}$, it is necessary to add another (abelian) factor $\mathfrak{t}$ to the product—this is discussed in [12] but will not be important here.

The representations $W(d_3)$, $W(d_2)$, $W(d_1)$, $W(a_2)$, and $W(a_1)$ generate $\widehat{R}_{\text{gen}}$ as an algebra [12, Prop. 6.1]. If $w\colon \widehat{R}_{\text{gen}} \to R$ specializes $\mathbb{F}^{\text{gen}}$ to a resolution $\mathbb{F}$, then the restriction of $w$ to $\bigoplus W(d_i)$ determines $w$ uniquely.

Given a map $w\colon \widehat{R}_{\text{gen}} \to R$ for a complex $(R, \mathbb{F})$, we denote by $w^{(i)}$ the restriction of $w$ to the representation $W(d_i) \subset R_{\text{gen}}$ and $w^{(a_i)}$ the restriction of $w$ to the representation $W(a_i)$. We typically view these maps as having source $R \otimes V_{-}(-\omega)$, e.g. we think of $w^{(3)}$ as a map

$$w^{(3)}\colon R \otimes V_{-}(-\omega_{z_{r-1}}) = R \otimes [F_3 \oplus M^* \otimes \bigwedge^{f_0+1} F_1 \oplus \cdots] \to R \otimes F_2.$$

We use $w_j^{(*)}$ to denote the restriction of $w^{(*)}$ to the $j$-th graded component of the representation, indexed so that $j = 0$ corresponds to the bottom graded piece. For instance, $w_0^{(i)} = d_i$ for $i = 1, 3$ and $w_0^{(2)} = d_2^*$. We call the maps $w_{>0}^{(*)}$ (a specific choice of) *higher structure maps* for $\mathbb{F}$. We refer the reader to [12, Example 3.3] for a demonstration of Theorem 2.3 in the context of higher structure maps.

2.2.6. *The ground field.* REWRITE

More precisely, the following construction via the Chevalley-Serre relations yields the "split form" of $\mathfrak{g}$.

One crucial detail that we have omitted until now is that Weyman constructed and studied $\widehat{R}_{\text{gen}}$ over $\mathbb{C}$, rather than $\mathbb{C}$. However, the construction of $\widehat{R}_{\text{gen}}$ and its decomposition into representations still make sense over $\mathbb{Q}$, and they are compatible with base-change from $\mathbb{Q}$ to $\mathbb{C}$. In other words, if we use $(\widehat{R}_{\text{gen}}, \mathbb{F}^{\text{gen}})$ to denote the objects as constructed here over $\mathbb{Q}$, then [19] and [18] study the pair $(\widehat{R}_{\text{gen}} \otimes_{\mathbb{Q}} \mathbb{C}, \mathbb{F}^{\text{gen}} \otimes_{\mathbb{Q}} \mathbb{C})$.

A lot of machinery is involved in proving the acyclicity of $\mathbb{F}^{\text{gen}} \otimes_{\mathbb{Q}} \mathbb{C}$ in [18]. One might be concerned about checking all of it over $\mathbb{Q}$ instead, but we point out two reasons why this will not pose an issue for us:

- A complex $\mathbb{F}$ over $\mathbb{Q}$ is acyclic if and only if $\mathbb{F} \otimes_{\mathbb{Q}} \mathbb{C}$ is. Thus the acyclicity of $\mathbb{F}^{\text{gen}}$ follows immediately from the acyclicity of $\mathbb{F}^{\text{gen}} \otimes_{\mathbb{Q}} \mathbb{C}$ already established.
- We will never actually use the acyclicity of $\mathbb{F}^{\text{gen}}$ at any point in this paper! This may come as a surprise, but notice that the classical Hilbert-Burch theorem does not actually claim the universal example to be acyclic—that is a separate result. Similarly, the first structure theorem of Buchsbaum and Eisenbud does not include the statement that $(R_a, \mathbb{F}^a)$ is universal for finite free complexes acyclic in grade 1. In essence, the fact that these theorems can be recast as "universal examples" is a certification that they are the *best possible* structure theorems for their respective objects. From this perspective, it is less surprising that $(\widehat{R}_{\text{gen}}, \mathbb{F}^{\text{gen}})$ has utility independent of the acyclicity of $\mathbb{F}^{\text{gen}}$.

2.3. **Schubert varieties.** Working over $\mathbb{C}$, there is a unique simply connected Lie group $G$ with Lie algebra $\mathfrak{g} \otimes \mathbb{C}$. The representations of $G$ correspond to those of $\mathfrak{g} \otimes \mathbb{C}$. For a fundamental weight $\omega_t$, the action of $G$ on the highest weight line in $\mathbb{P}(V(\omega_t))$ has stabilizer $P_t^+$, the subgroup of $G$ corresponding to the maximal parabolic subalgebra $\mathfrak{p}_t^+ \otimes \mathbb{C}$ as defined in §2.1.2. Hence the orbit of this highest weight line can be identified with the homogeneous space $G/P_t^+$. For Dynkin type $A_n$ with the standard labeling of vertices, this construction produces the Grassmannian $\text{Gr}(t, n+1)$. Accordingly, the reader may think of $G/P_t^+$ as a "generalized Grassmannian."

In the interest of only introducing what is necessary for discussing our results over $\mathbb{C}$, we will not take this perspective and we will instead define all the objects we need completely algebraically. However, we will use notation which alludes to the geometric construction. For instance we will define the scheme "$G/P_t^+$" via its homogeneous coordinate ring, without actually defining $G$, the subgroup $P_t^+$, or what it means to take a quotient. In a similar vein we will also define notions such as "$P_t^-$-orbits," although we will include some remarks to motivate such terminology.

That being said, we will occasionally want to use legitimate group actions, but the group in each case will either be the special or general linear group (c.f. Remark 2.1), or the exponential of some Lie algebra whose elements have locally nilpotent actions (so that the exponential is algebraically well-defined).

2.3.1. *Definition of the homogeneous space $G/P$.* Pick a vertex $t \in T$, let $\omega_t$ be the corresponding fundamental weight, and $V(\omega_t)$ the irreducible representation with highest weight $\omega_t$. Let $V_-(-\omega_t)$ be the irreducible representation with lowest weight $-\omega_t$; since these representations are finite-dimensional we have $V(\omega_t)^* = V_-(-\omega_t)$. Let $\mathfrak{A}$ be the $\mathbb{C}$-algebra

$$\bigoplus_{n\geq 0} V_-(-n\omega_t) = (\operatorname{Sym} V_-(-\omega_t))/I_{\text{Plücker}}$$

where, viewing $\operatorname{Sym} V_-(-\omega_t)$ as polynomial functions on $V(\omega_t)$, the ideal $I_{\text{Plücker}}$ is comprised of subrepresentations vanishing on a highest weight vector $v \in V(\omega_t)$. As an ideal it is generated by quadrics, namely the irreducible representations in $S_2 V_-(-\omega_t)$ other than $V_-(-2\omega_t)$. These are the *Plücker relations.*

The ring $\mathfrak{A}$ is graded with $V_-(-\omega_t)$ in degree 1. We define

$$G/P_t^+ := \operatorname{Proj} \mathfrak{A} \subset \mathbb{P}(V(\omega_t)).$$

**Remark 2.5.** By taking just the $\mathbb{C}$-points of this scheme, one recovers the object described at the beginning of this section.

2.3.2. *Weyl group and subgroups.* Let $W$ denote the Weyl group associated to $T$. It is generated by the *simple reflections* $\{s_i\}_{t\in T}$. Explicitly,

$$W = \langle \{s_i\}_{i\in T} \mid (s_i s_j)^{m_{ij}} = 1\rangle$$

where $m_{ij} = 1$ if $i = j$, $m_{ij} = 2$ if $i, j \in T$ are not adjacent, and $m_{ij} = 3$ if $i, j \in T$ are adjacent. Since we are assuming $T$ to be a Dynkin diagram, $W$ is finite.

The Weyl group acts on $\mathfrak{h}^*$: the simple reflections act via

$$s_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda\rangle \alpha_i.$$

**Remark 2.6.** It is also possible to lift (non-uniquely) each element $\sigma \in W$ to the group $G$ mentioned in Remark 2.1. We can circumvent dealing with the group $G$ by noting that each individual simple reflection $s_t$ can be lifted to an element of the group $SL_2$ corresponding to the Lie algebra $\mathfrak{sl}_2$ spanned by $e_t, f_t, \alpha_t^\vee$. This group acts on $\mathfrak{g}$ and its representations $V(\lambda)$ for $\lambda$ dominant. We can get an action of any $\sigma \in W$ by writing it as a product of simple reflections.

We will often abuse notation and write e.g. $\sigma v$ with $\sigma \in W$ and $v \in V(\lambda)$ when we really mean to pick a lift of $\sigma$ and then act by that on $v$. One explicit choice lifting $s_t$ is $\exp(f_t)\exp(-e_t)\exp(f_t)$, though the particular choice will not matter for our purposes.

The *length* $\ell(\sigma)$ of an element $\sigma \in W$ is the minimum number of simple reflections needed to express $\sigma$ as a product of simple reflections. There is a unique longest element $w_0 \in W$.

If $t \in T$, we let $W_{P_t} \subset W$ denote the subgroup generated by all simple reflections other than $s_t$. Every coset $W/W_{P_t}$ has a minimal length representative, and the set of such representatives is denoted by $W^{P_t}$.

2.3.3. *Plücker coordinates and Schubert cells.* Let $v \in V(\omega_t)$ be a highest weight vector, and $[v] \in \mathbb{P}(V(\omega_t))$ its span. The subgroup $W_{P_t}$ stabilizes $[v]$, so the extremal weights in $\mathbb{P}(V(\omega_t))$ are $\sigma[v]$ for $\sigma \in W^{P_t}$. Let $k$ be a field of characteristic zero. The $k$-points of $G/P_t^+$ decompose into a disjoint union of *Schubert cells*

$$(G/P_t^+)(k) = \coprod_{\sigma \in W^{P_t}} C^\sigma(k)$$

where $C^\sigma(k)$ is the orbit of the $k$-point $\sigma[v]$ under the action of $\exp(\mathfrak{n}^- \otimes k)$. For $j \in T$ and $[\sigma] \in W_{P_j}\backslash W/W_{P_t}$, we say that the union of Schubert cells

$$\coprod_{\sigma' \in W^{P_t}, [\sigma'] = [\sigma]} C^\sigma(k)$$

is a $P_j^-$*-orbit*, where the union is taken over $\sigma'$ representing the same double coset as $\sigma$. See also Remark 2.7.

The variables of $\mathfrak{A}$ are called *Plücker coordinates*. Let $p_e$ denote a lowest weight vector of $V_-(-\omega_t)$ and for $\sigma \in W^{P_t}$ let $p_\sigma = \sigma p_e$. The set $\{p_\sigma : \sigma \in W^{P_t}\}$ is the set of *extremal Plücker coordinates*. The representation $V_-(-\omega_t)$ may have weights other than those in the $W$-orbit of $-\omega_t$; these remaining Plücker coordinates are called non-extremal. (The representation is *miniscule* if all weights belong to the same $W$-orbit, in which case all Plücker coordinates are extremal.)

The Plücker coordinate $p_\sigma$ vanishes on the Schubert cell $C^w(k)$ iff $\sigma \not\leq w$ in the Bruhat order.

2.3.4. *The Schubert variety $X^w$.* Using the same notation as above, let $\mathfrak{n}^- wv$ denote the $\mathfrak{n}^-$-representation generated by $wv$ inside of $V(\omega_t)$. This is a *Demazure module*. Let $I(X^w)$ be the ideal of $\mathfrak{A}$ generated by the elements of $V_-(-\omega_t) = V(\omega_t)^*$ which vanish on $\mathfrak{n}^- wv$. The Schubert variety $X^w$ is the subscheme cut out by $I(X^w)$.

We will only be interested in a very specific Schubert variety, so we will now specialize the discussion accordingly. For the remainder of §2.3, we will always assume that $p = 2$ and $t = x_1$, so that the distinguished node is the only node on the left arm. Let $w = s_{z_1} s_u s_{x_1}$; it is a minimal length representative for its coset in $W/W_{P_{x_1}}$. We define $X^w$ as above. The Plücker coordinates which vanish on $\mathfrak{n}^- wv \subset V(\omega_{x_1})$ are exactly those in the bottom $z_1$-graded component $F_1 \subset V_-(-\omega_{x_1})$ (c.f. §2.1.2 for explanation of $F_i$). Explicitly these are the following $q+2$ coordinates, all of which are extremal:

$$p_e, p_{s_{x_1}}, p_{s_u s_{x_1}}, \ldots, p_{s_{y_{q-1}} \cdots s_{y_1} s_u s_{x_1}}.$$

All Schubert varieties are by construction preserved under the exponential action of $\mathfrak{n}^-$, but this one is moreover preserved under the action of $SL(F_1) \times SL(F_3)$ corresponding to the subdiagram $T - \{z_1\}$. Hence it is preserved under the action of the semidirect product $\exp(\mathfrak{n}_{z_1}^-) \rtimes (SL(F_3) \times SL(F_1))$

**Remark 2.7.** This semidirect product is not exactly equal to the group $P_{z_1}^-$ whose definition we have omitted, but it results in the same orbits on $G/P_{x_1}^+$, and thus will suffice as a substitute for our purposes. The Schubert variety $X^w$ can also be described as the complement of the open $P_{z_1}^-$-orbit.

2.3.5. *Affine patches.* The scheme $G/P_t^+$ is covered by the open sets $\sigma C^e := \{p_\sigma \neq 0\}$ for $\sigma \in W^{P_t}$. Each of these is isomorphic to affine space of dimension $\ell(w_0^{P_t})$ where $w_0^{P_t} \in W^{P_t}$ is a minimal length representative of $[w_0] \in W/W_{P_t}$. This is the dimension of $\mathfrak{n}_t^-$, the negative part of $\mathfrak{g}$ in the $t$-grading, and this algebra can be used to explicitly identify $\sigma C^e$ with affine space as follows.

Let $S$ be the polynomial ring $\mathrm{Sym}(\mathfrak{n}_t^-)^*$ and let

$$Z \in \mathfrak{n}_t^- \otimes (\mathfrak{n}_t^-)^* \subset \mathfrak{n}_t^- \otimes S$$

be adjoint to the identity on $\mathfrak{n}_t^-$. We think of $Z$ as the "generic element" of $\mathfrak{n}_t^-$. Then $\sigma C^e$ may be parametrized as

$$S \xrightarrow{i_{x_1}^{\mathrm{top}}} S \otimes V(\omega_{x_1}) \xrightarrow{\sigma \exp Z} S \otimes V(\omega_{x_1})$$

where $i_{x_1}^{\mathrm{top}}$ denotes the inclusion of the top $x_1$-graded component.

2.3.6. *The coordinate ring of $X^w$ restricted to an open cell.* Combining §2.3.5 with §2.3.4 and using the same notation, we get that the entries of the composite

$$d_1^*\colon S \xrightarrow{i_{x_1}^{\mathrm{top}}} S \otimes V(\omega_{x_1}) \xrightarrow{\sigma \exp Z} S \otimes V(\omega_{x_1}) \xrightarrow{p_{z_1}^{\mathrm{top}}} S \otimes F_1^*$$

give the equations cutting out $X^w$ restricted to $\sigma C^e$, where $p_{z_1}^{\mathrm{top}}$ denotes the projection onto the top $z_1$-graded component. Defining $d_1$ to be the dual of this composite, its cokernel is the coordinate ring of $X^w \cap \sigma C^e$. Moreover, it is shown in [13] how to extend this to a free resolution: define $d_2$ to be

$$d_2\colon S \otimes F_2 \xrightarrow{i_{x_1}^{\mathrm{top}}} S \otimes V(\omega_{y_d}) \xrightarrow{\sigma \exp Z} S \otimes V(\omega_{y_d}) \xrightarrow{p_{z_1}^{\mathrm{top}}} S \otimes F_1$$

and $d_3$ to be the dual of

$$d_3^*\colon S \otimes F_2^* \xrightarrow{i_{x_1}^{\mathrm{top}}} S \otimes V(\omega_{z_t}) \xrightarrow{\sigma \exp Z} S \otimes V(\omega_{z_t}) \xrightarrow{p_{z_1}^{\mathrm{top}}} S \otimes F_3^*.$$

**Theorem 2.8.** *The maps defined above assemble into a resolution of the coordinate ring of $X^w \cap \sigma C^e$:*

$$0 \to S \otimes F_3 \xrightarrow{d_3} S \otimes F_2 \xrightarrow{d_2} S \otimes F_1 \xrightarrow{d_1} S.$$

*Proof.* The paper [13] actually only discussed the case $\sigma = w_0$, in which case applying $w_0$ and then projecting onto the top $z_1$-graded component is the same as projecting onto the bottom $z_1'$-graded component, where $z_1'$ is the node on $T$ "dual" to $z_1$ if $T$ has exceptional duality (c.f. [13, Remark 2.2]). That is the manner in which the differentials are presented in the referenced paper. But this was just a matter of perspective and not the result of any technical limitation, as we will discuss briefly around Theorem 4.10. The proof of [13, Lemma 2.4] works exactly the same, showing that this is a complex. For acyclicity, the proof of [13, Theorem 5.1] shows how one can find powers of the Plücker coordinates among the minors of $d_i$. The only difference is that these are now written in terms of the affine coordinates on the patch $\sigma C^e$ instead of $w_0 C^e$, but that is inconsequential to the proof. □

Localizing this at the "origin" $\sigma[v]$ of the affine patch $\sigma C^e$, corresponding to the ideal of variables in $S$, gives a resolution of the local ring $\mathcal{O}_{X^w,\sigma[v]}$ over $\mathcal{O}_{G/P_t^+,\sigma[v]}$.

## 3. Surjectivity of the maps $w^{(i)}$

In this section we will prove the main technical result of this paper, which underlies the classification in subsequent sections.

**Theorem 3.1.** *Suppose we have a free resolution*

$$\mathbb{F}\colon 0 \to F_3 \to F_2 \to F_1 \to F_0$$

*of Dynkin format* $\underline{f} = (f_0, f_1, f_2, f_3)$ *over a* $\mathbb{C}$*-algebra* $R$*, whose dual* $0 \to F_0^* \to F_1^* \to F_2^* \to F_3^*$ *is also acyclic. Let* $\widehat{R}_{\text{gen}}(\underline{f})$ *be the generic ring associated to the format* $\underline{f}$*, and* $w\colon \widehat{R}_{\text{gen}}(\underline{f}) \to R$ *a map specializing* $\mathbb{F}^{\text{gen}}$ *to* $\mathbb{F}$*. Then the structure maps*

$$w^{(3)}\colon R \otimes V_-(-\omega_{z_{r-1}}) = R \otimes [F_3 \oplus M_0^* \otimes \bigwedge^{f_0+1} F_1 \oplus \cdots] \to R \otimes F_2$$

$$w^{(2)}\colon R \otimes V_-(-\omega_{y_{q-1}}) = R \otimes [F_1^* \oplus M_0^* \otimes F_3^* \otimes \bigwedge^{f_0} F_1 \oplus \cdots] \to R \otimes F_2^*$$

$$w^{(1)}\colon R \otimes V_-(-\omega_{x_{p-1}}) = R \otimes [F_1 \oplus M_0^* \otimes F_3^* \otimes \bigwedge^{f_0+2} F_1 \oplus \cdots] \to R \otimes F_0$$

$$w^{(a_3)}\colon R \otimes V_-(-\omega_{z_1}) = R \otimes [\bigwedge^{f_3} F_3 \oplus \cdots] \to R \otimes \bigwedge^{f_3} F_2$$

$$w^{(a_2)}\colon R \otimes V_-(-\omega_{x_1}) = R \otimes [\bigwedge^{r_2} F_1^* \otimes \bigwedge^{f_3} F_3^* \oplus \cdots] \to R \otimes \bigwedge^{f_2} F_2^*$$

*are surjective.*

Note that with the assumptions of the theorem, the Buchsbaum-Eisenbud multiplier $a_1$ yields an isomorphism

$$a_1\colon M = \bigwedge^{f_3} F_3 \otimes \bigwedge^{f_2} F_2^* \otimes \bigwedge^{f_1} F_1 \xrightarrow{\cong} \bigwedge^{f_0} F_0 = M_0.$$

Using this identification, we have replaced the tensor powers of $M^*$ appearing in the graded decomposition of the critical representations by powers of $M_0^*$. If $f_0 = 1$ (i.e. $p = 2$), then this also identifies the map $w^{(1)}$ with $w^{(a_2)}$. In the later sections where we classify perfect ideals, we will primarily make use of the surjectivity of $w^{(1)}$. As it is a map to $R$, its surjectivity is equivalent to being nonzero modulo the maximal ideal $\mathfrak{m} \subset R$.

The prototypical example of Theorem 3.1 mentioned in the introduction is $w^{(2)}$ for $\underline{f} = (1, n, n, 1)$. This is a $n \times 2n$ matrix consisting of the differential $d_2$ and an isomorphism $F_1 \cong F_2^*$ induced by a choice of multiplication on $\mathbb{F}$. The surjectivity of the matrix is evident from the presence of an invertible submatrix.

However for other Dynkin formats, it is hard to generalize this method to find an invertible submatrix. Instead, we will prove surjectivity of the maps $w^{(*)}$ by exhibiting them inside of larger invertible matrices. We give a brief sketch of the idea; the rest of this section will be devoted to the details. Since $\mathbb{F}^*$ is also acyclic, there is a map $w'\colon \widehat{R}_{\text{gen}}(\underline{f}') \to R$ where $\underline{f}' = (f_3, f_2, f_1, f_0)$ is the dual format, specializing the generic resolution to $\mathbb{F}^*$. From this homomorphism, one obtains structure maps $w'^{(*)}$ for $\mathbb{F}^*$. After dualizing, these have the form

$$(w'^{(3)})^*\colon R \otimes F_1 \to R \otimes [F_0 \oplus M_3^* \otimes \bigwedge^{f_3+1} F_2 \oplus \cdots] = R \otimes V_-(-\omega_{y_{q-1}})$$

$$(w'^{(2)})^*\colon R \otimes F_1^* \to R \otimes [F_2^* \oplus M_3^* \otimes F_0^* \otimes \bigwedge^{f_3} F_2 \oplus \cdots] = R \otimes V_-(-\omega_{y_{q-1}})$$

$$(w'^{(1)})^*\colon R \otimes F_3 \to R \otimes [F_2 \oplus M_3^* \otimes F_0^* \otimes \bigwedge^{f_3+2} F_2 \oplus \cdots] = R \otimes V_-(-\omega_{x_{p-1}})$$

$$(w'^{(a_3)})^*\colon R \otimes \bigwedge^{f_0} F_1 \to R \otimes [\bigwedge^{f_0} F_0 \oplus \cdots] = R \otimes V_-(-\omega_{z_1})$$

$$(w'^{(a_2)})^*\colon R \otimes \bigwedge^{f_1} F_1^* \to R \otimes [\bigwedge^{r_2} F_2^* \otimes \bigwedge^{f_0} F_0^* \oplus \cdots] = R \otimes V_-(-\omega_{x_1})$$

Each of these matrices has its first "block" in common with one of the maps $w^{(*)}$:

- $d_3$ appears in $w^{(3)}$ and $(w'^{(1)})^*$,
- $d_2$ appears in $w^{(2)}$ and $(w'^{(2)})^*$,
- $d_1$ appears in $w^{(1)}$ and $(w'^{(3)})^*$,
- $a_3$ appears in $w^{(a_3)}$ and $a_1 \otimes (w'^{(a_2)})^*$,
- $a_2$ appears in $w^{(a_2)}$ and $a_1^{-1} \otimes (w'^{(a_3)})^*$.

Our goal is to find invertible matrices $A_i$, $i \in \{z_{r-1}, y_{q-1}, x_{p-1}, z_1, x_1\}$, containing each of the above pairs of structure maps. For example, writing $V = V_-(-\omega_{x_{p-1}})$, we will construct $A_{x_{p-1}} \in \mathrm{Aut}_R(V \otimes R)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Aut}_R(V \otimes R) \arrow[rrr, "\text{project onto bottom } F_0"] \arrow[ddd, "\text{restrict to bottom } F_1"'] & & & \mathrm{Hom}_R(V \otimes R, F_0 \otimes R) \arrow[ddd] \\
& A_{x_{p-1}} \arrow[r, mapsto] \arrow[d, mapsto] & w^{(1)} \arrow[d, mapsto] & \\
& (w'^{(3)})^* \arrow[r, mapsto] & d_1 & \\
\mathrm{Hom}_R(F_1 \otimes R, V \otimes R) \arrow[rrr] & & & \mathrm{Hom}_R(F_1 \otimes R, F_0 \otimes R)
\end{tikzcd} \tag{3.1}$$

From this, the surjectivity of $w^{(1)}$ immediately follows. The approach for the other structure maps is completely analogous.

The construction of these $A_i$ is motivated by the action of $G$ (the group corresponding to the Dynkin type under consideration) on the representation $V_-(-\omega_i)$. We will construct the five automorphisms $A_i$ simultaneously. First we do this over a localization of $R$ where $\mathbb{F}$ becomes split exact, e.g. $R_h$ for any $h \in I_{f_0}(d_1)$. Then we show that the constructed automorphisms agree over different such localizations. Since grade $I_{f_0}(d_1) = 3 \geq 2$, we may find a regular sequence $h_1, h_2 \in I_{f_0}(d_1)$. The preceding implies that each $A_i$ is an automorphism over $R_{h_1} \cap R_{h_2} = R$, and this concludes the proof.

### 3.1. **The split exact case.**

The starting observation is that if we take each $A_i$ to be the identity, then in the bottom right of (3.1), we get the differentials of a split exact complex. We will refer to this complex as the *standard split complex* $\mathbb{F}_{\mathrm{ssc}}$. Note that it comes equipped with a splitting: the subdiagram of $T$ consisting of the center node and the right arm corresponds to $\mathfrak{sl}(C) \subset \mathfrak{g}$ where $C = \mathbb{C}^{r_2}$. Here $r_i$ is the rank of the differential $d_i$, e.g. $r_2 = f_2 - f_3$. Then $F_2 = F_3 \oplus C$ and $F_1 = F_0 \oplus C$.

Moreover, the top right and bottom left of (3.1) give structure maps for $\mathbb{F}_{\mathrm{ssc}}$ and $\mathbb{F}^*_{\mathrm{ssc}}$ as well. This is the content of [12, Theorem 4.5]. In other words, taking each $A_i$ to be the identity on the respective representation $V_-(-\omega_i)$ suffices for $\mathbb{F}_{\mathrm{ssc}}$ with the easy choice of higher structure maps $w_{\mathrm{ssc}}\colon \widehat{R}_{\mathrm{gen}}(\underline{f}) \to R$ and $w'_{\mathrm{ssc}}\colon \widehat{R}_{\mathrm{gen}}(\underline{f}') \to R$ from [12, Theorem 4.5]. For instance, we have

$$\begin{tikzcd}
\mathrm{Id}_{V_-(-\omega_{x_{p-1}})} \arrow[r, mapsto] \arrow[d, mapsto] & w^{(1)} \text{ for } \mathbb{F}_{\mathrm{ssc}} \text{ from } w_{\mathrm{ssc}} \arrow[d, mapsto] \\
(w'^{(3)})^* \text{ for } \mathbb{F}^*_{\mathrm{ssc}} \text{ from } w'_{\mathrm{ssc}} \arrow[r, mapsto] & d_1 \text{ of } \mathbb{F}_{\mathrm{ssc}}
\end{tikzcd} \tag{3.2}$$

and analogous diagrams for the other four pairs.

Now suppose $\mathbb{F}$ is a split exact complex over some ring $R$. Then one can find an isomorphism $\mathbb{F}_{\mathrm{ssc}} \otimes R \cong \mathbb{F}$, which amounts to picking appropriate $R$-valued points $g_1, g_2$ of $GL(F_1)$ and $GL(F_2)$:

$$\begin{tikzcd}
0 \arrow[r] & F_3 \otimes R \arrow[r] \arrow[d, equal] & F_2 \otimes R \arrow[r] \arrow[d, "g_2"] & F_1 \otimes R \arrow[r] \arrow[d, "g_1"] & F_0 \otimes R \arrow[d, equal] \\
0 \arrow[r] & F_3 \otimes R \arrow[r, "d_3"] & F_2 \otimes R \arrow[r, "d_2"] & F_1 \otimes R \arrow[r, "d_1"] & F_0 \otimes R
\end{tikzcd}$$

Explicitly $g_1, g_2$ are such that

$$g_1^{-1} = \begin{array}{c c} & F_1 \\ \begin{array}{c} F_0 \\ C \end{array} & \begin{bmatrix} d_1 \\ \gamma \end{bmatrix} \end{array} \qquad g_2 = \begin{array}{c c} & \begin{array}{cc} F_3 & C \end{array} \\ F_2 & \begin{bmatrix} d_3 & \beta \end{bmatrix} \end{array}$$

with the property that the composite $F_1 \xrightarrow{\gamma} C \xrightarrow{\beta} F_2$ splits the differential $d_2$.

We act on (3.2) by $g_1$ and $g_2$, as well as on the maps $w_{\mathrm{ssc}}\colon \widehat{R}_{\mathrm{gen}}(\underline{f}) \to R$ and $w'_{\mathrm{ssc}}\colon \widehat{R}_{\mathrm{gen}}(\underline{f'}) \to R$ from [12, Theorem 4.5]. Note that the actions of $GL(F_j)$ for different $j$ commute, so the order does not matter. Afterwards, (3.2) for example becomes

$$\begin{tikzcd}
\text{element of } \mathrm{Aut}_R(V_-(-\omega_{x_{p-1}}) \otimes R) \arrow[r, mapsto] \arrow[d, mapsto] & w^{(1)} \text{ for } \mathbb{F} \text{ from } w_0 \arrow[d, mapsto] \\
(w'^{(3)})^* \text{ for } \mathbb{F}^* \text{ from } w'_0 \arrow[r, mapsto] & d_1 \text{ of } \mathbb{F}
\end{tikzcd} \tag{3.3}$$

while $w_{\mathrm{ssc}}$ and $w'_{\mathrm{ssc}}$ become maps $w_0$ and $w'_0$ respectively, which give the structure maps in the above diagram (and the four other analogous diagrams).

Now suppose that we have already chosen higher structure maps for $\mathbb{F}$ and $\mathbb{F}^*$, corresponding to homomorphisms $w\colon \widehat{R}_{\mathrm{gen}}(\underline{f}) \to R$ and $w'\colon \widehat{R}_{\mathrm{gen}}(\underline{f'}) \to R$. There is a way to relate $w_0$ and $w'_0$ to $w$ and $w'$, given by [12, Theorem 3.1], as follows. Let $\mathbb{L}$ denote the negative part of $\mathfrak{g}$ in the $z_1$-grading and let $\mathbb{L}'$ denote the positive part of $\mathfrak{g}$ in the $x_1$-grading. These are the defect Lie algebras for $\mathbb{F}$ and $\mathbb{F}^*$. There exists a unique element $Z_- \in \mathbb{L} \otimes R$ such that precomposing $w_0$ by the action of $\exp Z_-$ on $R \otimes \widehat{R}_{\mathrm{gen}}(\underline{f})$ results in $w$. So if we precompose by $\exp Z_-$ in (3.3), we adjust the structure maps for $\mathbb{F}$ to the desired ones. Note that this action not affect the structure maps for $\mathbb{F}^*$, since for those maps the domain was already restricted to the bottom $z_1$-graded piece.

Similarly, there exists a unique element $X_+ \in \mathbb{L}' \otimes R$ such that acting on $w'_0$ by $\exp X_+$ yields $w'$. Thus postcomposing (3.3) by $\exp X_+$ adjusts the structure maps for $\mathbb{F}^*$ to the desired ones, without affecting the structure maps for $\mathbb{F}$. After these adjustments, the diagram (3.3) becomes

$$\begin{tikzcd}
A_{x_{p-1}} \arrow[r, mapsto] \arrow[d, mapsto] & w^{(1)} \text{ for } \mathbb{F} \text{ from } w \arrow[d, mapsto] \\
(w'^{(3)})^* \text{ for } \mathbb{F}^* \text{ from } w' \arrow[r, mapsto] & d_1 \text{ of } \mathbb{F}
\end{tikzcd} \tag{3.4}$$

where $A_{x_{p-1}}$ is our desired element of $\mathrm{Aut}_R(V_-(-\omega_{x_{p-1}}) \otimes R)$, and similarly for the other $A_i$.

Let us write

$$\alpha_i \colon G \to \operatorname{Aut} V_-(-\omega_i)$$
$$\sigma_i \colon GL(F_1) \to \operatorname{Aut} V_-(-\omega_i)$$
$$\rho_i \colon GL(F_2) \to \operatorname{Aut} V_-(-\omega_i)$$

for the actions of $G$, $GL(F_1)$, and $GL(F_2)$ on the representation $V_-(-\omega_i)$. With this notation, we can thus summarize the construction of each $A_i$ as follows:

$$A_i = \alpha_i(\exp X_+)\rho_i(g_2)\sigma_i(g_1^{-1})\alpha_i(\exp Z_-). \tag{3.5}$$

By construction, $A_i$ is evidently invertible.

3.2. **Independence of choice of splitting.** Given a split exact complex $\mathbb{F}$ and choices of higher structure maps for $\mathbb{F}$ and $\mathbb{F}^*$, to construct the matrices $A_i$ in the preceding subsection, we relied on the choice of a particular isomorphism $\mathbb{F}_{\mathrm{ssc}} \otimes R \cong \mathbb{F}$. This was the only step that required a choice; the elements $Z_-, X_+$ were uniquely determined afterwards.

We now show that the automorphisms obtained in the end are actually insensitive to this choice. In the following we will often abuse notation and just write e.g. $F_j$ when we mean $F_j \otimes R$.

**Lemma 3.2.** *Suppose that we pick a different isomorphism $\mathbb{F}_{\mathrm{ssc}} \otimes R \cong \mathbb{F}$, or equivalently, a different splitting $F_1 \xrightarrow{\gamma'} C \xrightarrow{\beta'} F_2$. Then there exist $\theta \in GL(C)$, $\eta_1 \in \operatorname{Hom}(F_0, C)$, and $\eta_2 \in \operatorname{Hom}(C, F_3)$ such that*

$$\gamma' = \theta\gamma + \eta_1 d_1, \quad \beta' = \beta\theta^{-1} + d_3\eta_2.$$

*For the corresponding $g_1', g_2'$, we can write this as*

$$g_1'^{-1} = \theta(1+\theta^{-1}\eta_1)g_1^{-1}, \quad g_2' = g_2(1+\eta_2\theta)\theta^{-1}$$

*recalling that $F_1 = F_0 \oplus C$ and $F_2 = F_3 \oplus C$.*

*Proof.* Both $\gamma, \gamma'$ must map $\ker d_1$ isomorphically onto $C$, so there exists an element $\theta \in GL(C)$ such that $\gamma' = \theta\gamma$ restricted to $\ker d_1$. The difference $\gamma' - \theta\gamma$ must then factor through $d_1$. This gives the first expression.

One similarly argues the existence of $\theta' \in GL(C)$ such that $\beta' = \beta\theta'$ modulo $\ker d_2 \subset F_2$. Note that if $s \colon F_1 \to F_2$ is a splitting, then

$$\ker d_1 \hookrightarrow F_1 \xrightarrow{s} F_2 \twoheadrightarrow F_2/(\ker d_2)$$

must be inverse to the map induced by $d_2$. In particular, $\beta\gamma$ and $\beta'\gamma'$ must agree as maps $(\ker d_1) \to F_2/(\ker d_2)$, which means $\theta' = \theta^{-1}$. The expression for $\beta'$ thus follows. $\square$

Note that $1+\theta^{-1}\eta_1 \in SL(F_1) \subset G$. It can be written as $\exp(\theta^{-1}\eta_1)$, viewing $\theta^{-1}\eta_1 \in F_0^* \otimes C = \mathfrak{g}_{1,0}$. Similarly $1+\eta_2\theta = \exp(\eta_2\theta)$ viewing $\eta_2\theta \in C^* \otimes F_3 = \mathfrak{g}_{0,-1}$.

If we go through the construction of §3.1 with $g_1', g_2'$, we get

$$A_i' = \alpha_i(\exp X_+')\rho_i(g_2')\sigma_i(g_1'^{-1})\alpha_i(\exp Z_-').$$

Expanding this using the above observations, we have

$$A_i' = \alpha_i(\exp X_+')\rho_i(g_2)\alpha_i(\exp(\eta_2\theta))\rho_i(\theta^{-1})\sigma_i(\theta)\alpha_i(\exp(\theta^{-1}\eta_1))\sigma_i(g_1^{-1})\alpha_i(\exp Z_-').$$

Now we use:

**Lemma 3.3.** *The map $GL(C) \to GL(F_1) \xrightarrow{\sigma_i} \operatorname{Aut}(V_i)$ agrees with $GL(C) \to GL(F_2) \xrightarrow{\rho_i} \operatorname{Aut}(V_i)$.*

*Proof.* The statement is certainly true for $SL(C)$ because both actions can be seen through the inclusion $SL(C) \subset G$. Note that in both cases, $GL(C)$ acts on each $(x_1, z_1)$-bigraded piece of the representation, and moreover if scalars $c \in GL(C)$ act by $c^\lambda$ on the part in bidegree $(m, n)$, then they act by $c^{\lambda+u+v}$ on the part in bidegree $(m+u, n+v)$.

Thus it is sufficient to check that the action of scalars $c \in GL(C)$ agree on a single bigraded component, such as the bottom one, and this is straightforward to verify from the explicit decompositions. We do $i = x_{p-1}$ as an example. The representation of $GL(F_1)$ in bottom $z_1$-degree is $F_1 = C \oplus F_3$, where $F_3$ resides in lower $x_1$-degree. So $c \in GL(C) \subset GL(F_1)$ acts by $c^0$ on the bottom bigraded component. Likewise, the bottom $x_1$-graded piece is $F_0$, a trivial representation of $GL(F_2)$, so $c \in GL(C) \subset GL(F_2)$ also acts by $c^0$. □

Hence $\rho_i(\theta^{-1})$ and $\sigma_i(\theta)$ cancel, and we are left with

$$A_i' = \alpha_i(\exp X_+')\rho_i(g_2)\alpha_i(\exp(\eta_2\theta))\alpha_i(\exp(\theta^{-1}\eta_1))\sigma_i(g_1^{-1})\alpha_i(\exp Z_-').$$

Elements of $\mathfrak{g}_{1,0}$ and $\mathfrak{g}_{0,-1}$ commute because $\mathfrak{g}_{1,-1} = 0$, so we can interchange the middle two terms. Note that

$$\begin{aligned}\theta^{-1}\eta_1 \in \mathfrak{g}_{1,0} &= F_0^* \otimes C \\ &\subset \mathfrak{g}_{1,*} = F_0^* \otimes \bigwedge^{f_3+1} F_2 \otimes \bigwedge^{f_3} F_3^*\end{aligned}$$

Applying $g_2$ to $\theta^{-1}\eta_1$ gives an element $X_1 \in \mathfrak{g}_{1,*}$ such that

$$\alpha_i(\exp X_1)\rho_i(g_2) = \rho_i(g_2)\alpha_i(\exp(\theta^{-1}\eta_1)).$$

Similarly, by applying $g_1$ to $\eta_2\theta$, we get $Z_1 \in \mathfrak{g}_{*,-1}$ such that

$$\sigma_i(g_1^{-1})\alpha_i(\exp Z_1) = \alpha_i(\exp(\eta_2\theta))\sigma_i(g_1^{-1}),$$

allowing us to write

$$A_i' = \alpha_i(\exp X_+')\alpha_i(\exp X_1)\rho_i(g_2)\sigma_i(g_1^{-1})\alpha_i(\exp Z_1)\alpha_i(\exp Z_-').$$

Baker-Campbell-Hausdorff yields elements $\widetilde{X}_+ \in \mathbb{L}'$ and $\widetilde{Z}_- \in \mathbb{L}$ such that

$$\exp \widetilde{X}_+ = (\exp X_+')(\exp X_1), \quad \exp \widetilde{Z}_- = (\exp Z_1)(\exp Z_-'),$$

and so

$$A_i' = \alpha_i(\exp \widetilde{X}_+)\rho_i(g_2)\sigma_i(g_1^{-1})\alpha_i(\exp \widetilde{Z}_-).$$

However, compare this to

$$A_i = \alpha_i(\exp X_+)\rho_i(g_2)\sigma_i(g_1^{-1})\alpha_i(\exp Z_-)$$

and recall that $X_+, Z_-$ were *uniquely* determined so that this expression would recover the chosen structure maps for $\mathbb{F}$ and $\mathbb{F}^*$ in (3.3). Thus $X_+ = \widetilde{X}_+$ and $Z_- = \widetilde{Z}_-$, from which we conclude that $A_i = A_i'$.

Having established that the matrices $A_i$ are independent of the choice of splitting, Theorem 3.1 readily follows.

*Proof of Theorem 3.1.* Let $\mathbb{F}$ be a resolution over $R$ of Dynkin format $\underline{f} = (f_0, f_1, f_2, f_3)$, and suppose that $\mathbb{F}^*$ is also acyclic. Fix higher structure maps for $\mathbb{F}$ and $\mathbb{F}^*$, i.e. maps $\widehat{R}_{\text{gen}}(\underline{f}) \to R$ and $\widehat{R}_{\text{gen}}(\underline{f}') \to R$ specializing the generic resolutions to $\mathbb{F}$ and $\mathbb{F}^*$ respectively.

Writing $d_1$ for the first differential of $\mathbb{F}$, let $h_1, h_2 \in I_{f_0}(d_1)$ be a regular sequence. Then $\mathbb{F} \otimes R_{h_1}$ and $\mathbb{F} \otimes R_{h_2}$ are split exact, so using the construction of §3.1, we obtain matrices $A_i$ over $R_{h_1}$ and $A_i'$ over $R_{h_2}$. Working over the common localization $R_{h_1 h_2}$, the results of this section imply that $A_i = A_i'$.

In particular, this means that the entries of each $A_i$ are elements of $R_{h_1} \cap R_{h_2} = R$. The same goes for $(\det A_i)^{-1}$. Thus the matrices $A_i$ are invertible over $R$, from which the surjectivity of the stated structure maps follows immediately. ☐

3.3. **First applications.** The major applications of Theorem 3.1 will be explored in later sections. We conclude this section with a easy but surprising consequence in the graded setting: the theorem gives a restriction on graded Betti numbers. Let $k$ be a field of characteristic zero, $R = k[x_0, \dots, x_m]$, and $M$ a graded Cohen-Macaulay $R$-module. Suppose the graded minimal free resolution of $M$ has the form

$$0 \to \bigoplus_{j=1}^{b_3} R(-s_{3j}) \to \bigoplus_{j=1}^{b_2} R(-s_{2j}) \to \bigoplus_{j=1}^{b_1} R(-s_{1j}) \to \bigoplus_{j=1}^{b_0} R(-s_{0j}) \to M \to 0.$$

**Corollary 3.4.** *In the setup above, if $(b_0, b_1, b_2, b_3)$ is Dynkin, then the graded module*

$$R \otimes V_-(-\omega_{x_1}) = \bigwedge^{f_2} F_2 \otimes [\bigwedge^{r_2} F_1^* \otimes \bigwedge^{f_3} F_3^* \oplus \cdots]$$

*must have a generator in degree zero.*

*Proof.* For a graded resolution, it is possible to pick $w\colon \widehat{R}_{\text{gen}} \to \mathbb{C} \otimes R$ such that the higher structure maps are all homogeneous of degree zero—this follows from the fact that the structure maps $p_i$ which determine $w$ are computed by recursive lifting. Having done this, Theorem 3.1 implies that $w^{(a_2)}$ is surjective, i.e. nonzero mod $\mathfrak{m} = (x_1, \dots, x_m)$. Hence one entry of the matrix must have degree exactly zero. ☐

One case of particular interest is $F_0 = R$, so $M = R/I$. As $a_1$ is an isomorphism, the above equivalently says that

$$R \otimes V_-(-\omega_{x_1}) = F_1 \oplus F_3^* \otimes \bigwedge^3 F_1 \oplus \cdots,$$

the domain of $w^{(1)}$, has a generator in degree zero. Each graded component is a $\prod GL(F_i)$-subrepresentation of $F_1 \otimes \mathfrak{g}_1^{\otimes j}$ for some $j \geq 0$, where $\mathfrak{g}_1 = F_3^* \otimes \bigwedge^2 F_1$.

**Example 3.5.** If $R/I$ is Cohen-Macaulay and $(1, b_1, b_2, b_3)$ is Dynkin, then $2\min(s_{1j}) < \max(s_{3j})$. Since the module $F_1$ is generated in positive degrees, it follows that $F_3^* \otimes \bigwedge^2 F_1$ must have a generator in negative degree in order for $R \otimes V_-(-\omega_{x_1})$ to have a generator in degree zero.

From the perspective of linkage, this example can also be obtained as a corollary of a result we will establish in a forthcoming paper, which is that every perfect ideal of grade 3 with Dynkin Betti numbers is licci. The inequality $2\min(s_{1j}) < \max(s_{3j})$ then follows from [10, Corollary 5.13].

**Example 3.6.** If $R/I$ is Cohen-Macaulay and $(1, b_1, b_2, b_3)$ is Dynkin, then at least one $s_{1j}$ is even or at least one $s_{3j}$ is odd. If this were not the case, $F_1$ would be generated in odd degree and $F_3^* \otimes \bigwedge^2 F_1$ would be generated in even degree. Consequently $R \otimes V_-(-\omega_{x_1})$ would have all generators in odd degree, thus none in degree zero.

## 4. Classification of perfect ideals with Dynkin Betti numbers

We will now apply Theorem 3.1 to classify perfect ideals $I$ of grade 3 in a local ring $(R, \mathfrak{m}, k)$ of equicharacteristic zero[3], with the assumption that $R/I$ has Betti numbers in the Dynkin range. If $R$

[3]As discussed in §2.2.6, although $\widehat{R}_{\text{gen}}$ was originally constructed over $\mathbb{C}$ in [19] and [18], higher structure maps can be computed for any resolution over a $\mathbb{Q}$-algebra. A local $\mathbb{Q}$-algebra is equivalently a local ring of equicharacteristic zero.

is regular, this is equivalent to $R/I$ being Cohen-Macaulay, with one of the following conditions on the type $t$ and deviation $d$:

- $t = 1$,
- $d = 1$,
- $t = 2$ and $d \leq 4$, or
- $t \leq 4$ and $d = 2$.

The first two cases are already well-understood. The novel results come from the last two—these correspond to Dynkin types $E_n$.

In order to do this, we will make use of an important connection between $\widehat{R}_{\text{gen}}$ and the homogeneous space $G/P^+_{x_1}$. We refer to §2.3 for background regarding the following setup. Throughout this whole section, we fix a Dynkin format $(1, f_1, f_2, f_3) = (1, 3+d, 2+d+t, t)$. Let $T_{2,d+1,t+1}$ be the corresponding diagram

$$\begin{tikzcd}[arrows=dash]
x_1 \arrow[r] & u \arrow[r] \arrow[d] & y_1 \arrow[r] & \cdots \arrow[r] & y_d \\
& z_1 \arrow[d] & & & \\
& \vdots \arrow[d] & & & \\
& z_t & & &
\end{tikzcd}$$

Let $V = V(\omega_{x_1})$, and let $v \in V$ be a highest weight vector. Write $[v] \in \mathbb{P}(V)$ for its span.

We now recall the relationship between $\widehat{R}_{\text{gen}}$ and $G/P^+_{x_1} \subset \mathbb{P}(V)$ given in [12]. Note that this can also be inferred from the explicit decompositions of $\widehat{R}_{\text{gen}}$ into representations of $\mathfrak{g}$ given in [18].

**Theorem 4.1.** *The subring generated by the representation $W(d_1) = V^*$ inside of $\widehat{R}_{\text{gen}}$ is the homogeneous coordinate ring $(\operatorname{Sym} V^*)/I_{\text{Plücker}}$ of $G/P^+_{x_1}$ in its Plücker embedding.*

Let $I \subset R$ be a perfect ideal of grade 3 such that $R/I$ is resolved by $\mathbb{F}$ of the format $(1, f_1, f_2, f_3)$. We do not require that $\mathbb{F}$ be minimal. Let $w\colon \widehat{R}_{\text{gen}} \to \mathbb{F}$ be a map specializing the generic resolution to $\mathbb{F}$. From Theorem 3.1 we know that $w^{(1)}$ is surjective, equivalently nonzero mod $\mathfrak{m}$, and hence Theorem 4.1 implies that we have a map $\operatorname{Spec} R \to G/P^+_{x_1}$. In particular, by looking at $w^{(1)} \otimes k$, where $k = R/\mathfrak{m}$ is the residue field, we get a $k$-point of $G/P^+_{x_1}$. Let $P^-_{z_1} \subset G$ correspond to the non-positive part of $\mathfrak{g}$ in the $z_1$-grading. The $k$-points of $G/P^+_{x_1}$ are a disjoint union of $P^-_{z_1}$-orbits, each of which is a union of $B^-$-orbits (i.e. Schubert cells). The next result says that the orbit containing $w^{(1)} \otimes k$ is well-defined and preserved under local specialization.

**Proposition 4.2.** *Suppose that $\mathbb{F}$ has Dynkin format and resolves $R/I$ for a perfect ideal $I \subset R$. Let $w\colon \widehat{R}_{\text{gen}} \to R$ specialize the generic resolution to $\mathbb{F}$. The $P^-_{z_1}$-orbit containing the $k$-point determined by $w^{(1)} \otimes k$ depends only on the ideal $I \subset R$ and not on the choice of resolution $\mathbb{F}$ or map $w\colon \widehat{R}_{\text{gen}} \to R$ specializing the generic resolution to $\mathbb{F}$.*

*If $I' \subset R'$ is a perfect ideal in regular local ring $(R', \mathfrak{m}', k')$ of equicharacteristic zero and $R \xrightarrow{\varphi} R'$ is a local homomorphism such that $\varphi(I)R' = I'$, then the orbits determined by $I$ and $I'$ are the same.*

*Proof.* For the first point, let $\mathbb{F}'$ be a different resolution of format $(1, f_1, f_2, f_3)$ for $R/I$, and $w'\colon \widehat{R}_{\text{gen}} \to R$ a map specializing the generic resolution to $\mathbb{F}'$. We view $w, w'$ as maps $R \otimes \widehat{R}_{\text{gen}} \to R$. By precomposing $w$ with the action of an appropriate element $g \in \prod GL(F_i)$ on $R \otimes \widehat{R}_{\text{gen}}$, we can arrange so that $wg\colon R \otimes \widehat{R}_{\text{gen}} \to R$ specializes the generic resolution to $\mathbb{F}'$. Since both $w'$ and $wg$ have this property, it follows from [12, Theorem 3.1] that there is an element $X \in \mathbb{L} \otimes R$, where $\mathbb{L}$ is the negative part of $\mathfrak{g}$ in the $z_1$-grading, such that $wg \exp X = w'$.

The action of $\prod GL(F_i)$ on $V$, up to a scalar, can be seen from the part of $G$ corresponding to the middle $z_1$-graded piece of $\mathfrak{g}$, which is $\mathfrak{sl}(F_1) \times \mathfrak{sl}(F_3) \times \mathbb{C}$. For more details we refer the reader to §2 or [12, §2.3]. Evidently $X \in \mathbb{L}$ is in the negative $z_1$-graded part. We conclude that both of these actions preserve the $P_{z_1}^-$ orbit that contains $w^{(1)} \otimes k$.

For the other point of the proposition, simply note that $\mathbb{F} \otimes R'$ is a resolution of $R'/I'$, and

$$(w^{(1)} \otimes_R R') \otimes_{R'} k' = (w^{(1)} \otimes_R k) \otimes_k k'$$

because we assumed $\varphi$ to be a local homomorphism, so it induces an inclusion of residue fields $k \hookrightarrow k'$. $\square$

Let $W$ denote the Weyl group of $G$ and, for $j \in T_{2,d+1,t+1}$, $W_{P_j} \subset W$ the subgroup generated by all simple reflections $\{s_i\}_{i \neq j}$. The Schubert cells of $G/P_{x_1}^+$ are indexed by the torus-fixed points. In the Plücker embedding $G/P_{x_1}^+ \hookrightarrow \mathbb{P}(V)$, these are exactly the extremal weight lines, which are in correspondence with $W/W_{P_{x_1}}$.

The $P_{z_1}^-$-orbits in $G/P_{x_1}^+$, each of which is a union of Schubert cells, are indexed by the double cosets $W_{P_{z_1}} \backslash W / W_{P_{x_1}}$. If $\sigma \in W$ is the minimal length representative of such a double coset, then $\sigma \cdot v \in V$ is a highest weight vector for an extremal representation of $\mathfrak{sl}(F_3) \times \mathfrak{sl}(F_1)$ inside of $V$. In this manner, the $P_{z_1}^-$-orbits correspond to these extremal representations, and we obtain the following algebraic translation of the above proposition, in the language of higher structure maps.

**Proposition 4.3.** *Suppose that $\mathbb{F}$ has Dynkin format and resolves $R/I$ for a perfect ideal $I \subset R$. Let $w\colon \widehat{R}_{\text{gen}} \to R$ specialize the generic resolution to $\mathbb{F}$. In the $z_1$-graded decomposition*

$$R \otimes V_-(-\omega_{x_1}) = F_1 \oplus F_3^* \otimes \bigwedge^3 F_1 \oplus \cdots$$

*there is a lowest irreducible $\mathfrak{gl}(F_3) \times \mathfrak{gl}(F_1)$-representation to which the restriction of $w^{(1)}$ is nonzero mod $\mathfrak{m}$. This representation depends only on $I \subset R$ and is necessarily extremal.*

To summarize, so far we have demonstrated how a perfect ideal determines an element of $W_{P_{z_1}} \backslash W / W_{P_{x_1}}$ which can be used to classify the ideal. Next we will show that every double coset is realizable in this manner, and exhibit a generic perfect ideal for each double coset.

**Theorem 4.4.** *Let $w = s_{z_1} s_u s_{x_1} \in W$ and let $X^w \subset G/P_{x_1}^+$ be the codimension 3 Schubert variety that is the closure of $B^- w \cdot [v]$. Let $\sigma \in W$ be a representative of a double coset in $W_{P_{z_1}} \backslash W / W_{P_{x_1}}$ and let $S_\sigma := \mathcal{O}_{G/P_{x_1}^+, \sigma\cdot[v]}$ be the local ring of $G/P_{x_1}^+$ at $\sigma \cdot [v]$; it is isomorphic to a polynomial ring over $\mathbb{Q}$ localized at its ideal of variables. Let $I_\sigma$ be the ideal of local defining equations of $X^w$ at that point. The ideal $I_\sigma$ is the unit ideal if $[\sigma] = [e]$ where $e \in W$ is the identity. Otherwise it is a perfect ideal of grade 3 in $S_\sigma$. It has the following properties:*

1. *If $w\colon \widehat{R}_{\text{gen}} \to S_\sigma$ specializes the generic resolution to a resolution of $S_\sigma/I_\sigma$, the point of $G/P_{x_1}^+$ determined by $w^{(1)} \otimes \mathbb{Q}$ is in $P_{z_1}^- \sigma \cdot [v]$, the $P_{z_1}^-$-orbit corresponding to the double coset $[\sigma]$.*
2. *If $R$ is a local ring of equicharacteristic zero, $w\colon \widehat{R}_{\text{gen}} \to R$ specializes the generic resolution to a resolution of $R/I$ for a perfect ideal $I$, and $w^{(1)} \otimes k$ is in the same $P_{z_1}^-$-orbit as $\sigma \cdot [v]$, then there exists a local homomorphism $\varphi\colon S_\sigma \to R$ such that $I = \varphi(I_\sigma)R$.*

*Proof.* Note that $X^w$ is the union of all $P_{z_1}^-$-orbits aside from the big open orbit $P_{z_1}^- \cdot [v]$. So $I_\sigma$ is the unit ideal when $[\sigma] = [e]$. If $[\sigma] \neq [e]$ then $\sigma \cdot [v] \in X^w$ and it is well-known that Schubert varieties are Cohen-Macaulay [14], so $I_\sigma \subset S_\sigma$ is a perfect ideal.

To prove (1), in view of Proposition 4.2, it suffices to construct one such $w$ and verify the statement. We will produce this $w$ using the action of $G$ on $\widehat{R}_{\text{gen}}$. We begin with the map $w_{\text{ssc}}\colon \widehat{R}_{\text{gen}} \to \mathbb{C}$ from [12, Theorem 4.5] that was also used in §3, and observe that its restriction $w_{\text{ssc}}^{(1)}\colon V^* \to \mathbb{C}$ yields a highest weight vector $v \in V$. Let $\mathfrak{n}_{x_1}^-$ be the negative part of $\mathfrak{g}$ in the $x_1$-grading. The affine patch $B^- \cdot [v] \subset G/P_{x_1}^+$ (the open Schubert cell) is the orbit of $[v]$ under the exponential action of $\mathfrak{n}_{x_1}^-$, thus it can be identified with $\operatorname{Spec}\operatorname{Sym}(\mathfrak{n}_{x_1}^-)^*$.

Next we produce a map $\widehat{R}_{\text{gen}} \to \operatorname{Sym}(\mathfrak{n}_{x_1}^-)^*$ by precomposing

$$\widehat{R}_{\text{gen}} \xrightarrow{w_{\text{ssc}}} \mathbb{C} \to \operatorname{Sym}(\mathfrak{n}_{x_1}^-)^*$$

with the action of $\exp X$ on $\widehat{R}_{\text{gen}} \otimes \operatorname{Sym}(\mathfrak{n}_{x_1}^-)^*$, where

$$X \in \mathfrak{n}_{x_1}^- \otimes (\mathfrak{n}_{x_1}^-)^* \subset \mathfrak{n}_{x_1}^- \otimes \operatorname{Sym}(\mathfrak{n}_{x_1}^-)^*$$

is the "generic element" of $\mathfrak{n}_{x_1}^-$, i.e. $X$ is adjoint to the identity on $\mathfrak{n}_{x_1}^-$. This construction is akin to the one used in the proof of [12, Theorem 5.1] for "generic higher structure maps."

Finally we precompose this map by the action of $\sigma^{-1} \in W$ (or more accurately, a representative of $\sigma^{-1}$ in $G$) on $\widehat{R}_{\text{gen}}$. Let $w\colon \widehat{R}_{\text{gen}} \to \operatorname{Sym}(\mathfrak{n}_{x_1}^-)^*$ be the result. By construction, $w^{(1)}\colon V^* \to \operatorname{Sym}(\mathfrak{n}_{x_1}^-)^*$ is none other than a parametrization of the open patch $\sigma \exp(\mathfrak{n}_{x_1}^-) \cdot [v]$, where $\sigma \cdot [v]$ is the origin, corresponding to the ideal of variables in $\operatorname{Sym}(\mathfrak{n}_{x_1}^-)^*$. In particular, $S_\sigma$ is the localization of this polynomial ring at its ideal of variables. The ideal $I_\sigma$ is generated by the Plücker coordinates coming from the bottom $z_1$-graded component $w_0^{(1)}$ of $w^{(1)}$.

The map $w$ specializes the generic resolution to a complex over $S_\sigma$ with differentials $w_0^{(1)}$, $w_0^{(2)}$, and $w_0^{(3)}$. This complex is none other than the resolution of $S_\sigma/I_\sigma$ given in Theorem 2.8.

Point (2) follows readily from the discussion before Proposition 4.2. If $w\colon \widehat{R}_{\text{gen}} \to R$ is such a map, and $p$ is the $k$-point of $G/P_{x_1}^+$ determined by $w^{(1)} \otimes k$, then the ideal of $X^w \subset G/P_{x_1}^+$ at the point $p$ specializes to the ideal $I$. The important point is that the action of $P_{z_1}^-$ on $G/P$ preserves the Schubert variety $X^w$. Consequently, since $p$ and $\sigma \cdot [v]$ are related by an element of $P_{z_1}^-$, the local defining equations of $X^w$ at these two points are equivalent up to a change of coordinates. Thus $I_\sigma$ specializes to $I$ as well. □

We will not explicitly describe the ideals $I_\sigma$ in this paper; they can get very complicated and there are simply too many of them for $E_7$ and $E_8$. The paper [13] outlines how to produce free resolutions of $S_\sigma/I_\sigma$ for $\sigma = w_0$ the longest element, and that construction is easily adapted to other $\sigma$.

However, we will at least tie back our results to the discussion in the introduction §1 and describe the situation for Dynkin types $D_n$ and $E_6$. Before doing so, we note that even without explicitly understanding the ideals $I_\sigma$, we obtain the following classification result as a corollary of the above.

**Theorem 4.5.** *Fix a Dynkin format $(1, f_1, f_2, f_3)$ and the corresponding setup as in the beginning of this section. Every element of $W_{P_{z_1}} \backslash W / W_{P_{x_1}} - [e]$ describes a non-empty family of perfect ideals of grade 3 with Betti numbers $(1, b_1, b_2, b_3)$, where $b_i \leq f_i$ for all i. These families are disjoint, and this is the finest possible classification that is preserved under local specialization.*

*Proof.* If $I \subset R$ is perfect of grade 3, the condition that $b_i \leq f_i$ is equivalent to saying that $R/I$ admits a (not necessarily minimal) resolution of format $(1, f_1, f_2, f_3)$, and so its classification comes from Proposition 4.2. The non-emptiness of each family comes from Theorem 4.4 point (1). Disjointness comes from Proposition 4.2, as does the fact that this classification is preserved under local specialization. The existence of a generic example for each family, Theorem 4.4 point (2), shows that it is the finest classification with this property. □

Now we revisit the types $D_n$ and $E_6$, starting with the two families of $D_n$ formats.

**Example 4.6.** Consider the format $(1,n,n,1)$ where $n \geq 3$. The corresponding diagram is $T_{2,n-2,2} = D_n$, and the representation $V$ is a half-spinor representation. The $z_1$-graded decomposition of $V^*$ into $\mathfrak{gl}(F_3) \times \mathfrak{gl}(F_1)$-representations is

$$V^* = (F_1) \oplus (F_3^* \otimes \bigwedge^3 F_1) \oplus (S_2 F_3^* \otimes \bigwedge^5 F_1) \oplus \cdots \oplus (S_{\lfloor \frac{n-1}{2} \rfloor} F_3^* \otimes \bigwedge^{2\lfloor \frac{n-1}{2} \rfloor + 1} F_1)$$

Every representation appearing is extremal; they correspond to the elements of $W_{P_{z_1}} \backslash W / W_{P_{x_1}}$. Aside from the lowest representation $F_1$, which corresponds to $[e] \in W_{P_{z_1}} \backslash W / W_{P_{x_1}}$, there are $\lfloor \frac{n-1}{2} \rfloor$ extremal representations; each one of these is a possible location for the lowest appearance of a unit in the structure map $w^{(1)}$.

If $I \subset R$ is a perfect ideal of grade 3 such that $R/I$ has Betti numbers $(1, b_1, b_2, b_3)$ with $b_i \leq f_i$, then necessarily $b_3 = 1$ and $b_1 = b_2 \leq n$. Gorenstein ideals of grade 3 are minimally generated by an odd number of elements [16]. Moreover, for each odd $b_1$ with $3 \leq b_1 \leq n$, the generic example of such an ideal is given by [3], confirming Theorem 4.5 for this format.

**Example 4.7.** Consider the format $(1,4,n,n-3)$ where $n \geq 4$. The corresponding diagram is $T_{2,2,n-2} = D_n$, and the representation $V$ is again a half-spinor representation. However, the node $z_1$ is different from the preceding example, and the decomposition of $V^*$ into $\mathfrak{gl}(F_3) \times \mathfrak{gl}(F_1)$-representations is

$$V^* = (F_1) \oplus (F_3^* \otimes \bigwedge^3 F_1) \oplus (\bigwedge^2 F_3^* \otimes S_{2,1^3} F_1) \oplus \cdots \oplus (\bigwedge^{n-3} F_3^* \otimes S_{(a+1)^b, a^{(4-b)}} F_1).$$

Here $a = \lfloor \frac{n-3}{2} \rfloor$, and $b = 2 + (-1)^n$. Every representation appearing is extremal. Again excluding the lowest representation $F_1$, we see $n-3$ possible representations for the lowest appearance of a unit in $w^{(1)}$. These correspond to the Betti numbers $(1,3,3,1)$ and $(1,4,b_2,b_2-3)$ where $5 \leq b_2 \leq n$. There is a generic perfect ideal with each of these Betti numbers; see [3] and [1].

We refer the reader to [9] for a detailed discussion of higher structure maps for the preceding two formats, including methods for computing $w_j^{(i)}$ explicitly via lifting. Next we turn our attention to the $E_6$ format.

**Example 4.8.** Consider the format $(1,5,6,2)$. The corresponding diagram is $T_{2,3,3} = E_6$, and the representation $V$ is the adjoint. The $z_1$-graded decomposition of $V^*$ into $\mathfrak{gl}(F_3) \times \mathfrak{gl}(F_1)$-representations is

$$V^* = (F_1) \oplus (F_3^* \otimes \bigwedge^3 F_1) \oplus \begin{bmatrix} (S_2 F_3^* \otimes \bigwedge^5 F_1) \\ \oplus (\bigwedge^2 F_3^* \otimes S_{2,1^3} F_1) \\ \oplus (\bigwedge^2 F_3^* \otimes \bigwedge^5 F_1) \end{bmatrix} \oplus (S_{2,1} F_3^* \otimes S_{2^2,1^3} F_1) \oplus (S_{2,2} F_3^* \otimes S_{2^4,1} F_1)$$

All representations, except for the $\bigwedge^2 F_3^* \otimes \bigwedge^5 F_1$ appearing in the middle, are extremal.

If $R/I$ has Betti numbers $(1, b_1, b_2, b_3)$ with $b_i \leq f_i$, then the Betti numbers can be one of $(1,3,3,1)$, $(1,5,5,1)$, $(1,4,5,2)$, or $(1,5,6,2)$. The first three have already been discussed in the preceding two examples; in particular there is a generic example for each one. This leaves *two* elements of $W_{P_{z_1}} \backslash W / W_{P_{x_1}}$ for the Betti numbers $(1,5,6,2)$, namely those corresponding to the last two representations $S_{2,1} F_3^* \otimes S_{2^2,1^3} F_1$ and $S_{2,2} F_3^* \otimes S_{2^4,1} F_1$.

If one explicitly computes the ideals $I_\sigma$ for these two cases, the two can be distinguished by whether the multiplication on $\mathrm{Tor}_1(S_\sigma/I_\sigma, \mathbb{Q})$ is nonzero. For the former example, the multiplication is nonzero, and the ideal is described in [1, Theorem 4.4]. This multiplication is zero for the latter, and that $I_\sigma$ is the ideal $J(t)$ in [6]. Its relationship to $E_6$ is discussed at length in that paper as well as in [13, §3.2]. Since the property of this Tor algebra multiplication being (non)zero is preserved under local specialization as discussed in §1, it can be used to distinguish between the corresponding two families of $(1,5,6,2)$ perfect ideals.

The double cosets $W_{P_{z_1}}\backslash W/W_{P_{x_1}}$ can be computed algorithmically, and we thank Witold Kraśkiewicz for providing us with Python code that does so. By looking at the cardinality of this set, a simple counting argument can be used to deduce the number of families of perfect ideals with Betti numbers corresponding to types $E_7$ and $E_8$. We have already witnessed the first point of the following theorem in Example 4.8.

**Theorem 4.9.** *In the sense of Theorem 4.5, there are:*
- *2 families of perfect ideals with Betti numbers* $(1,5,6,2)$,
- *7 families of perfect ideals with Betti numbers* $(1,6,7,2)$,
- *11 families of perfect ideals with Betti numbers* $(1,5,7,3)$,
- *49 families of perfect ideals with Betti numbers* $(1,7,8,2)$,
- *90 families of perfect ideals with Betti numbers* $(1,5,8,4)$.

*Proof.* In Table 2, we have summarized the cardinality of $W_{P_{z_1}}\backslash W/W_{P_{x_1}}$ for various Dynkin formats $(1, 3+d, 2+d+t, t)$; call this quantity $\#(d,t)$.

TABLE 2. Cardinality $\#(d,t)$ of $W_{P_{z_1}}\backslash W/W_{P_{x_1}}$ for $T_{2,d+1,t+1}$ associated to Dynkin formats $(1, 3+d, 2+d+t, t)$

| $\#(d,t)$ | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ | $\cdots$ |
|---|---|---|---|---|---|---|
| $d=0$ | 2 | 2 | 2 | 2 | 2 | $\cdots$ |
| $d=1$ | 2 | 3 | 4 | 5 | 6 | $\cdots$ |
| $d=2$ | 3 | 6 | 18 | 109 | – | |
| $d=3$ | 3 | 13 | – | – | – | |
| $d=4$ | 4 | 63 | – | – | – | |
| $d=5$ | 4 | – | – | – | – | |
| $\vdots$ | $\vdots$ | | | | | |

Since $\#(d,t)-1$ records the number of families of perfect ideals with Betti numbers *at most* $(1,3+d,2+d+t,t)$, we have that the number of families with Betti numbers *exactly* the given format is

$$\#(d,t) - \#(d-1,t) - \#(d,t-1) + \#(d-1,t-1).$$

The counts in the theorem follow easily. □

We close this section by mentioning a non-local form of Theorem 4.4, to address the "genericity conjecture" that has appeared in previous work on the subject, e.g. [17, Questions 4.9].

In [13], the focus was on $\sigma = w_0 \in W$, the longest element. The reason for focusing on $w_0$ was a matter of perspective, rather than any technical limitation. Rather than looking at the local defining equations of $X^w$ at any particular point, both that paper and its precursor [15] examined the ideal of $Y^w := X^w \cap w_0C^e$ inside of $w_0C^e$, where $C^e = B^- \cdot [v]$ is the big open Schubert cell and $w_0C^e$ is its opposite. This yields an ideal $J_{w_0} \subset S \coloneqq \operatorname{Sym}(\mathfrak{n}_{x_1}^-)^*$. The "origin" $w_0 \cdot [v]$ of this open cell $w_0 C^e$ is a point in the lowest-dimensional $P_{z_1}^-$-orbit of $G/P_{x_1}^+$. So by localizing at the ideal of variables, we recover the ideal $I_{w_0} \subset S_{w_0}$.

Loosely speaking, the genericity conjecture says that a general choice of higher structure maps for a perfect ideal with Dynkin Betti numbers will have a unit in the top coordinate of $w^{(1)}$. This follows easily from Theorem 3.1, but it carries less information than Theorem 4.4. We present and prove it mainly for the sake of closing the loop in this circle of ideas.

**Theorem 4.10.** *Fix a Dynkin format $(1, f_1, f_2, f_3)$. If $\mathbb{F}$ is a resolution of $R/I$ with the given format, where $I \subset R$ is a perfect ideal of grade 3 in a local ring of equicharacteristic zero, then there is a map $\varphi: S \to R$ such that $\varphi(J_{w_0})R = I$.*

Note that the polynomial ring $\operatorname{Sym}(\mathfrak{n}_{x_1}^-)^*$ is *not* localized, as we are considering the whole affine patch $w_0 C^e$. Since $J_{w_0}$ and $I$ are both perfect ideals of grade 3, the resolution for $S/J_{w_0}$ constructed in [13] specializes to one for $R/I$ via $\varphi$. This is the form in which the statement appears in [13, Conjecture 2.6]. On the other hand, [17] states the genericity conjecture in terms of the "generic top complex" for a split exact complex. These apparently different formulations are reconciled in [12, Theorem 5.3].

*Proof.* Pick a map $w: \widehat{R}_{\mathrm{gen}} \to R$ specializing the generic resolution to $\mathbb{F}$, and let $w^{(1)}$ denote its restriction to $W(d_1) = V^*$ as usual.

Let $\lambda$ be a highest weight vector of $V^*$, i.e. dual to $w_0 \cdot v \in V$. Note that $\ker(w^{(1)} \otimes k)$ is a hyperplane in $V^*$ since $w^{(1)} \otimes k \neq 0$. The linear span of the $B^-$-orbit of $\lambda$ is the entirety of $V^*$, thus a general element $g \in B^-$ has the property that $g \cdot \lambda \notin \ker(w^{(1)} \otimes k)$.

Precomposing $w^{(1)}$ by such a $g$ does not change the image of $w_0^{(1)}$ because the bottom $z_1$-graded component $F_1 \subset V^*$ is preserved under the action of $B^-$. Hence $w^{(1)} g$ determines a map $\operatorname{Spec} R \to G/P_{x_1}$ landing in $w_0 C^e$, and the corresponding map $S \to R$ specializes $J_{w_0}$ to $I$ as desired. ☐

## 5. Next steps

The theory of $\widehat{R}_{\mathrm{gen}}$ has been developed for arbitrary resolution formats $(f_0, f_1, f_2, f_3)$ of length 3, not just those in the Dynkin range. Actually, the construction of $A_i$ in the proof of Theorem 3.1 works fine without the Dynkin hypothesis, the caveat being that they must be thought of as maps

$$A_i: V(\omega_i) \otimes V_-(-\omega_i) \to R.$$

Note that $V_-(-\omega_i)$ is the graded dual of $V(\omega_i)$, which is different from the ordinary dual if the representation is infinite. In particular, such a map $A_i$ cannot be interpreted as an endomorphism of $V_-(-\omega_i)$, and thus the crucial final step of the proof, in which we argue that the $A_i$ are invertible, does not make sense.

Phrasing it in this manner, one might be led to think that this is a technical shortcoming of the proof. However, the theorem statement itself is not even true without the Dynkin hypothesis, as the following example shows.

**Example 5.1.** Let $I = (x, y, z)^2 \subset R = \mathbb{C}[x, y, z]$. If we take $R$ with the standard grading, then $R/I$ admits a graded minimal free resolution

$$\mathbb{F}: 0 \to R(-4)^3 \to R(-3)^8 \to R(-2)^6 \to R.$$

The format $(1, 6, 8, 3)$ is not Dynkin, as it is associated to the affine type $\widetilde{E}_7$. The quotient $R/I$ is obviously Cohen-Macaulay, being zero-dimensional, so $\mathbb{F}^*$ is acyclic. Thus the other assumption of Theorem 3.1 is still satisfied.

However, we know that there exists $\widehat{R}_{\text{gen}} \to R$ resulting in $w^{(1)}$ being homogeneous of degree zero; see for instance the discussion around Corollary 3.4. The module

$$R \otimes V_{-}(-\omega_{x_1}) = F_1 \oplus F_3^* \otimes \bigwedge^3 F_1 \oplus \cdots$$

has all generators in degree 2, since $F_1$ is generated in degree 2 and $F_3^* \otimes \bigwedge^2 F_1$ is generated in degree 0. We conclude that the image of $w^{(1)}$ is $I$, so in particular $w^{(1)}$ is not surjective. This is because the entries of $w^{(1)}$ are all quadrics by degree considerations, and the differential $d_1$ is part of the matrix $w^{(1)}$.

In a future paper, we will elucidate a connection between higher structure maps coming from $\widehat{R}_{\text{gen}}$ and the theory of linkage. From that perspective, the surjectivity of $w^{(1)}$ is equivalent to the ideal $I$ being in the linkage class of a complete intersection (licci), and Theorem 3.1 will be recast as the following:

**Theorem 5.2.** *Let $I$ be a grade 3 perfect ideal in a local Noetherian ring $R$ of equicharacteristic zero. Let $d$ denote the deviation of $I$ and $t$ the minimal number of generators of* $\operatorname{Ext}^3(R/I, R)$. *If*

- *$d \leq 4$ and $t \leq 2$, or*
- *$d \leq 2$ and $t \leq 4$,*

*then $I$ is in the linkage class of a complete intersection.*

On the other hand, the ideal $(x, y, z)^2$ from the preceding example is *not* licci. Moreover, in [7] it is shown that for all $(1, f_1, f_2, f_3)$ outside the Dynkin range, there exists a perfect ideal with those Betti numbers that is not licci. Thus the Dynkin condition in Theorem 3.1 is essential.

Beyond the Dynkin range, it remains unclear how to use representation theory to characterize non-licci perfect ideals. A concrete starting point would be to see whether one can produce some well-known examples of non-licci perfect ideals with Betti numbers $\underline{f} = (1, 6, 8, 3)$ directly from the representation theory of $\widetilde{E}_7$, which is the affine Kac-Moody Lie algebra involved in the construction of $\widehat{R}_{\text{gen}}(\underline{f})$. Two particularly simple examples of such perfect ideals are

- the ideal of $2 \times 2$ minors of a generic $2 \times 4$ matrix (the ideal of $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ in the Segre embedding), and
- the ideal of $2 \times 2$ minors of a generic $3 \times 3$ symmetric matrix (the ideal of $\mathbb{P}^2 \subset \mathbb{P}^5$ in the Veronese embedding).

Next we discuss a different avenue for future work. The behavior of perfect ideals of codimension $c$ often parallels the behavior of Gorenstein ideals of codimension $c + 1$. Indeed, there are various methods of producing the latter given an example of the former. From this perspective, after studying perfect ideals of codimension 3, a natural next step is to examine Gorenstein ideals of codimension 4.

For a such an ideal $I$ generated by $n$ elements, and $\mathbb{F}$ a self-dual resolution of $R/I$, there is a ring $A(n)_\infty$ and a map $A(n)_\infty \to R$ which can be viewed as a collection of "higher structure maps" for $\mathbb{F}$. This construction is described in (need citation for Gorenstein codim 4). The Kac-Moody Lie algebra associated to the diagram $T_{3,n,2}$ acts on $A(n)_\infty$. In particular, when $6 \leq n \leq 8$, this Lie algebra is $E_n$. In analogy with Theorem 3.1, we conjecture that the higher structure maps, suitably interpreted, are surjective for $n \leq 8$. Assuming this, one can develop a similar classification as we have done in §4.

However, for the proof of Theorem 3.1, it was necessary to have *two* rings $\widehat{R}_{\text{gen}}(\underline{f})$ and $\widehat{R}_{\text{gen}}(\underline{f}')$ to assemble the matrices $A_i$. In a way, it would appear that the ring $A(n)_\infty$ only provides half of the picture needed to carry out this same program. For example, Kustin's "higher order products" are not visible in $A(n)_\infty$, and we expect to find them coming from another ring. We do not yet have a systematic construction of this second set of structure maps in general, but for $n = 6$, they have been explicitly computed via lifting (citation for Lorenzo's paper with the explicit liftings).

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