Committee: David Eisenbud (Advisor), Ian Agol (Chair), David Nadler, Mark Haiman

1. Major topic: Algebraic Geometry (Geometry)

References: Hartshorne's *Algebraic Geometry*, I.1-7, II.1-8, III.1-6, IV.1-5.

- **Schemes and morphisms** Affine, projective, reduced, irreducible, regular, and Noetherian schemes. Fiber products, varieties, and blowups. Open and closed embeddings, affine, finite, finite-type, separated, proper, projective, rational, and dominant morphisms. Valuative criteria.
- **Sheaves** Presheaves and sheaves. Quasicoherent, coherent, locally free, invertible, ample, very ample, and twisting sheaves. Relationship between Weil divisors, Cartier divisors, line bundles, and maps to $\mathbb{P}^n$. Sheaves of differentials.
- **Cohomology** Derived functor cohomology, Čech cohomology, $H^m(\mathbb{P}^n, \mathcal{O}(d))$, Grothendieck's vanishing theorem, Serre's criterion for affineness, Serre duality (statement).
- **Curves** Riemann-Roch, Hurwitz, embeddings of curves, elliptic curves, Clifford's theorem.

2. Major topic: Algebraic Topology (Geometry)

References: May's *A Concise Course in Algebraic Topology*, Ch. 1-23.

- **(Co)homology** Singular and cellular (co)homology, Eilenberg-Steenrod axioms, Tor and Ext, universal coefficient and Künneth theorems, cup and cap products, Poincaré duality, mod 2 Steenrod algebra.
- **Homotopy theory** Fundamental group, covering spaces, van Kampen, free groups and graphs. Higher homotopy groups, excision and suspension, LES of a fibration, Whitehead's theorem. Postnikov and Whitehead towers.
- **(Co)homology and homotopy** CW complexes, cellular approximation, $K(G, n)$, BG and the functors they represent, Hurewicz's theorem, Serre spectral sequence.
- **Characteristic classes** Vector bundles, Stiefel-Whitney, Euler, Chern, and Pontryagin classes, Grassmannians.

3. Minor topic: Lie Theory (Algebra)

References: Serre's *Complex Semisimple Lie Algebras*.

- Nilpotent, solvable, and semisimple Lie algebras. Theorems of Engel, Lie, and Ado. Cartan and Borel subalgebras. Root systems, Weyl groups, Dynkin diagrams. Universal enveloping algebras, Poincaré-Birkhoff-Witt, highest weight modules. Representation theory of $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{sl}_3(\mathbb{C})$. Correspondence between Lie algebras and groups.

I took my exam on April 21, 2020. A recount of it starts on the next page. It may not be entirely faithful in regard to the order of questions, and I may have forgotten some questions. My exam lasted two hours: 12:30PM to 2:30PM.

Algebraic Geometry

Eisenbud "Tell us about the cohomology of line bundles on $\mathbb{P}^n_k$."

Me All line bundles are isomorphic to $\mathcal{O}(d)$ for some integer d. Sum them all up and compute the cohomology all at once. I drew the following diagram, in which $S = k[x_0, \ldots, x_n]$ and $M = S^{\oplus(n+1)}$.

$$\begin{tikzcd}
 & 0 \arrow[r] & \prod S_{x_i} \arrow[r] & \prod S_{x_i x_j} \arrow[r] & \cdots \arrow[r] & S_{x_0\cdots x_n} \arrow[r] & 0 \\
 & & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \\
0 \arrow[r] & S \arrow[r, "{(x_0^2,\ldots,x_n^2)}"] & M \arrow[r, "{-\wedge(x_0^2,\ldots,x_n^2)}"] \arrow[u] & \wedge^2 M \arrow[r] \arrow[u] & \cdots \arrow[r] & \wedge^{n+1} M \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & S \arrow[r, "{(x_0,\ldots,x_n)}"] & M \arrow[r, "{-\wedge(x_0,\ldots,x_n)}"] \arrow[u, "{(x_0^2,\ldots,x_n^2)}"] & \wedge^2 M \arrow[r] \arrow[u] & \cdots \arrow[r] & \wedge^{n+1} M \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The bottom rows are Koszul complexes associated to regular sequences and thus their homology is concentrated at the far right only. Since direct limit is exact, it commutes with taking homology, in particular the original Cech complex has cohomology only at the extremes: S at the far left and $(x_0 \cdots x_n)^{-1} k[x_0^{-1}, \ldots, x_n^{-1}]$ at the far right. If you want the answer for $\mathcal{O}(d)$, just take the degree d part. (Note that the direct limit of $M \xrightarrow{f} M \xrightarrow{f} \cdots$ is M_f.)

Haiman "That's very algebraic; can you explain geometrically what these line bundles are?"

Me I briefly talked about how $\mathcal{O}(d)$ relates to a degree d hypersurface.

Nadler "Aside from the Cech complex computation, is there some other result you could use to deduce the cohomology of line bundles on projective space?"

Me I pondered this for a bit and stated that I could use Serre duality but it doesn't help me figure out that the middle cohomology groups are zero. (He was just fishing for Serre duality though, so he was content.)

Eisenbud "Can you compute $H^1(\Omega_{\mathbb{P}}(1))$?"

Me I wrote down the relevant exact sequence

$$0 \to \Omega^1(1) \to \mathcal{O}^{\oplus(n+1)} \to \mathcal{O}(1) \to 0$$

and then the beginning of the LES of cohomology groups. I stared at $H^0(\mathcal{O})^{\oplus(n+1)} \to H^0(\mathcal{O}(1))$ for a bit, worried because it was surjective but I was expecting the answer to be nonzero. That's because I was thinking about Ω^1 rather than $\Omega^1(1)$.

Eisenbud "Looks surjective to me!" He then also commented on how $\Omega^1(1)$ was the universal sub-bundle on $\mathbb{P}^n$. (The surjection $\mathcal{O}^{\oplus(n+1)} \to \mathcal{O}(1)$ corresponds to the identity morphism on $\mathbb{P}^n$.)

Eisenbud "Can you tell us about the relationship between line bundles, Cartier divisors, and Weil divisors? An example where they're different?"

Me I explained how the Cartier class group, being the cokernel of

$$H^0(\mathscr{K}^\times) \to H^0(\mathscr{K}^\times/\mathcal{O}^\times)$$

necessarily injects into the next term in the LES, which is none other than the Picard group $H^1(\mathscr{O}^\times)$. I said I don't know an example of it being non-surjective, and that such an example would necessarily have to be non-integral and non-projective.

I went on to say how there's a map from the Cartier class group to the Weil class group, and how it is injective but not surjective for the cone over a plane conic, while it is surjective but not injective for a nodal cubic. I wrote $\mathbb{G}_m \to 0$ and nobody commented on that, but of course that's only about the degree 0 parts. Though Weil divisors are not even defined for the nodal cubic in Hartshorne's text...

Eisenbud "I suppose we should ask something about curves. What can you say about a degree 4 curve in $\mathbb{P}^3$?"

Me First bound the genus ≤ 2 by considering a (necessarily singular, by linear normality considerations) projection to $\mathbb{P}^2$. But $g \neq 2$ because the lowest degree embedding a genus 2 curve has is degree 5. If $g = 0$ and it's nondegenerate then it's a twisted quartic. If $g = 1$ and it's nondegenerate, apply RR to the rightmost term below to conclude that there are at least two linearly independent quadrics containing the curve:

$$0 \to H^0(\mathscr{I}_X(2)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_X(2)).$$

The intersection of the two is one-dimensional of degree 4, so it's our curve.

Eisenbud "What theorem are you using to conclude that?"

Me I admitted I didn't know.

Eisenbud "It's the unmixedness theorem." That's how we know there's no embedded components.

Algebraic Topology

Agol "Let's discuss the same question as before in the topological setting. What can you say about complex line bundles on a space?"

Me I talked about how homotopy classes of maps into $BU(1)$ classifies line bundles, and how this is none other than $\mathbb{CP}^\infty$. I commented on how line bundles are special in the sense that they are fully classified by their one non-vanishing Chern class, because the classifying space is an Eilenberg-MacLane space.

Agol "Are there any hypotheses you'd like to impose on the base space?" (There might have been another question that led to this one, but I don't remember...)

Me The blanket assumption that I'd like to always impose when talking about vector bundles is that the base space is paracompact. This is needed so that pullbacks of a vector bundle along homotopic maps are isomorphic.

Agol "Can you compute the cohomology ring of $\mathbb{CP}^n$?"

Me I commented that I would take this opportunity to demonstrate that I know a little about spectral sequences. As I wrote down $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ and got started...

Nadler "Okay, but how do you *actually* think about the cup product in this setting?"

Me Similarly to the Chow ring in algebraic geometry; i.e. via intersections. (He said that he would be very worried if the spectral sequence was my main way of understanding the cohomology ring!) I won't typeset the spectral sequence computation I did here, but I arrived at the answer. At one point I remarked that I would still have to separately argue that $\mathbb{CP}^n$ is simply connected for this spectral sequence computation to be valid, e.g. by noting it has no 1-cells.

Agol "How else could you see that from what you've written?"

Me I demonstrated it using the LES of the fibration $S^1 \to S^{2n+1} \to \mathbb{CP}^n$.

Nadler "Do you know how to compute $H^*(\Omega S^3)$?"

Me I can do it essentially in the same way via the path-space fibration $\Omega S^3 \to PS^3 \to S^3$ where $PS^3 \simeq *$. (That's what he was looking for.) Without actually doing the computation, I stated that since $H^*(S^3)$ was exterior on a generator in degree 3, the answer was polynomial on a generator in degree 2. I realized after the exam that this is not correct; it should be a divided power algebra. He didn't correct me so I guess we were both thinking about $\mathbb{Q}$-coefficients.

Nadler "This exterior and polynomial algebra business—there's a mathematician's name that is closely associated to it, and it's come up already. Do you know who it is?"

Me I stared at the fibration I just used and guessed Serre, to which Nadler remarked "well, it would be wrong to say any of this *isn't* Serre!" and then talked a bit about Koszul duality.

Agol "Can you tell us about BG and what it represents? Can you give us some examples of BG for various G?"

Me I talked about principal G-bundles, and the relationship with vector bundles. I constructed a few examples of universal bundles $G \to EG \to BG$. For instance: if G is any finite group, I can put it in Σ_n and act on the configuration space of n labeled points in $\mathbb{R}^\infty$, etc. At one point in this, I was asked the following, since S^∞ is a frequent ingredient in constructing EG:

Nadler "Why is S^∞ contractible?"

Me There are at least two ways to see this. One is that it's a weakly contractible CW complex, thus actually contractible by Whitehead. Another is an actual concrete contraction—I demonstrated the "Eilenberg swindle" where I homotoped the identity to the right-shift map. The image of the right-shift map misses $(1, 0, \ldots)$ for example, so now you can contract away from that point.

Agol "If G is abelian, is BG again a group?"

Me I drew a diagram with the adjoint functors Π and N (fundamental groupoid and nerve) between groupoids and simplicial sets, and the adjoint functors $|-|$ and Sing (geometric realization and singular simplicial set) between simplicial sets and spaces.
One way to construct BG is as the geometric realization of the nerve. The significance of G being abelian is that multiplication $G \times G \to G$ is actually a group homomorphism, so I can apply $B(-)$ to it. I obtain $B(G \times G) \to BG$, whereas applying $B(-)$ to the projections $G \times G \to G$ and putting them together gives me $B(G \times G) \to BG \times BG$. So I have

$$BG \times BG \leftarrow B(G \times G) \to BG$$

which is almost what I want. So now there's a miracle, which is that the left map is actually a homeomorphism. I call it a miracle because while the nerve functor is a right adjoint and thus automatically preserves limits, geometric realization is a *left* adjoint so there's (at least to me) no obvious reason why it should preserve this product, yet it does.

Nadler "Along those lines, Π is a *left* adjoint in your diagram. There's a item on your syllabus related to what you just said—what is it?"

Me That would be the van Kampen theorem.

Agol "If A is an abelian group, can you rewrite $K(A, n)$ in terms of B?"

Me $B^n A$. I justified this by showing, from the LES for $A \to EA \to BA$, why B "deloops" a space, shifting all homotopy groups up by one.

I was also asked something about Poincaré duality in there somewhere (by Agol I think?), and Nadler asked me to revisit the nodal cubic I drew when we were talking about algebraic geometry. I laughed and said that I was prepared for this one, because he told me it was his favorite space (N: "Maybe we skip this question then..."). It's homotopy equivalent to $S^1 \vee S^2$, and fails Poincaré duality since the cup product is zero on middle cohomology.

Lie Theory

Haiman "Consider a 3-dimensional Lie algebra with basis X, Y, Z such that Z is central and $[X, Y] = Z$. Can you find a Lie group with this Lie algebra?"

Me I wasn't really sure how to systematically proceed with this, so...

Haiman "Can you say what kind of Lie algebra this is?"

Me My first observation was that it has a center so it's evidently not semisimple. After a bit of thought I realized that $[\mathfrak{g}, \mathfrak{g}] = \mathbb{C}Z$, and the next term in the lower central series is zero, so $\mathfrak{g}$ is nilpotent. That led me to the natural candidate

$$\begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}$$

with

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Haiman "What does this have to do with SL_3?"

Me It's not quite the Borel. (H: "How is it related?") It's the commutator.

I don't remember what exactly was asked next, but after this we went on to discuss representation theory of $\mathfrak{sl}_3(\mathbb{C})$. I drew the weight picture in $\mathfrak{h}^*$ for the adjoint representation, pointed out the positive roots, renamed X, Y, Z from the preceding question to $X_{1,2}, X_{2,3}, X_{1,3}$ respectively.

Haiman "So your answer to my first question gives a 3-dimensional representation of $\mathfrak{n}$. Can you draw the weight picture for the corresponding $\mathfrak{sl}_3$ representation?"

Me That's the standard representation. I drew the weight picture.

Haiman "How about the tensor square? Can you decompose it into irreducibles?"

Me I drew the weight picture and decomposed it as $\mathrm{Sym}^2 \mathbb{C}^3 \oplus \wedge^2 \mathbb{C}^3$, and also commented that the latter is also the dual of the standard rep.

Haiman "Are these irreducible?" (Me: "I'm pretty sure they are.") "How do you know?"

Me I fumbled around for a bit and gave some ad-hoc argument that I don't quite remember. Haiman gave another argument using highest weights.

Haiman "This isn't on your syllabus, but you can also do this problem using characters. Do you know how?"

Me I stated that I knew the Weyl character formula existed, but that I didn't remember it.

Nadler "Can you tell us about Verma modules for $\mathfrak{sl}_3$?"

Me I defined the Verma module $I_\lambda = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda$ for $\lambda \in \mathfrak{h}^*$.

Nadler "What are its weight multiplicities?"

Me I answered this visually, by indicating some paths I could take from λ to a given weight. (N: "How would you prove it?") PBW.
Then I explained how they could be used to resolve finite-dimensional irreducible representations of $\mathfrak{sl}_3$, which led to...

Haiman “Now you can derive the Weyl character formula!”

Me I wrote the resolution

$$\cdots \to \bigoplus_{l(w)=1} I_{w\cdot\lambda} \to I_\lambda \to V \to 0$$

where sums are over elements of the Weyl group with the specified length, and $w\cdot\lambda$ means the shifted reflection $w(\lambda+\rho)-\rho$, where ρ is 1/2 the sum of the positive roots. From this I wrote

$$ch(V) = \sum_{w\in W} (-1)^{l(w)} ch(I_{w(\lambda+\rho)-\rho})$$

and after much struggling and assistance wrote down

$$ch(I_{w(\lambda+\rho)-\rho}) = \sum_{k_1,k_2,k_3\in\mathbb{Z}_{\geq 0}} e^{\lambda-k_1\alpha_1-k_2\alpha_2-k_3\alpha_3}.$$

where the α_i are the positive roots. (I had been trying to write the sum in a way that there was one term for each weight, with some coefficient, but of course it's much simpler to just have one term for each way of getting there!)

Haiman “You could even rewrite this more compactly, since it's a geometric series.”

Me At this point I was really close to the character formula, but it seems like my brain gave out first because I stared at that expression for a minute completely forgetting how exponents work. So I just said “sorry I think I don't have enough mental clarity for this right now.”

They laughed and we ended the qual there.