Resonances as Viscosity Limits for Exponentially Decaying Potentials

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Abstract

We show that the complex absorbing potential (CAP) method for computing scattering resonances applies to the case of exponentially decaying potentials. That means that the eigenvalues of $-\Delta + V - i\varepsilon x^2$, $|V(x)| \leq Ce^{-2\gamma|x|}$ converge, as $\varepsilon \to 0+$, to the poles of the meromorphic continuation of $(-\Delta + V - \lambda^2)^{-1}$ uniformly on compact subsets of Re $\lambda > 0$, Im $\lambda > -\gamma$, arg $\lambda > -\pi/8$.

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1. INTRODUCTION

The complex absorbing potential (CAP) method has been used as a computational tool for finding scattering resonances – see Riss–Meyer [1] and Seideman–Miller [2] for an early treatment and Jagau et al [3] for some recent developments. For potentials $V \in L_{\text{comp}}^{\infty}$ the method was justified by Zworski [4]. In [5] the author extended it to potentials which are dilation analytic near infinity. In this paper we show that the CAP method is also valid for potentials which are exponentially decaying. While the key component of [4] and [5] was the method of complex scaling (see Hunziker [6], Sjöstrand–Zworski [7] for an account and references), here we use complex scaling on the Fourier transform side following Nakamura [8] and Kameoka–Nakamura [9].

Thus, we consider the Schrödinger operator $P := -\Delta + V$ acting on $L^2(\mathbb{R}^n)$ whose potential is exponentially decaying, this means that there exist $C, \gamma > 0$ such that

$$|V(x)| \le Ce^{-2\gamma|x|}.\tag{1.1}$$

Let $R_V(\lambda) = (P - \lambda^2)^{-1}$ be the resolvent of P, initially defined for $\text{Im } \lambda > 0$. The exponentially weighted resolvent $\sqrt{V}R_V(\lambda)\sqrt{V}$ can be meromorphically continued to the strip $\text{Im } \lambda > -\gamma$, see Froese [10], Gannot [11] and a review in §2. Resonances of P are the poles in this meromorphic continuation.

We now introduce a *regularized* operator,

$$P_{\varepsilon} := -\Delta - i\varepsilon x^2 + V, \quad \varepsilon > 0. \tag{1.2}$$

(We write $x^2 := x_1^2 + \cdots + x_n^2$.) It is easy to see, with details reviewed in §4, that P_{ε} is a non-normal unbounded operator on $L^2(\mathbb{R}^n)$ with a discrete spectrum. When $V \equiv 0$, P_{ε} is reduced to the rescaled Davies harmonic oscillator – see §3, whose spectrum is given by

$$\sqrt{\varepsilon} e^{-i\pi/4} (2|\alpha| + n), \quad \alpha \in \mathbb{N}_0^n, \quad |\alpha| := \alpha_1 + \dots + \alpha_n,$$

where \mathbb{N}_0 denotes the set of nonnegative integers. Thus we will restrict our attentions to arg $z > -\pi/4$. Suppose that

$$\sigma(P_{\varepsilon}) \cap \mathbb{C} \setminus e^{-i\pi/4}[0,\infty) = \{\lambda_j(\varepsilon)^2\}_{j=1}^{\infty}, \quad -\pi/8 < \arg \lambda_j(\varepsilon) < 7\pi/8.$$
(1.3)

Zworski [4] proved that resonances can be defined as the limit points of $\{\lambda_j(\varepsilon)\}_{j=1}^{\infty}$ as $\varepsilon \to 0+$, in the case of compactly supported potentials. We generalize this result to the case of exponentially decaying potentials. More precisely, we have **Theorem 1.** For any 0 < a' < a < b and $\gamma' < \gamma$ such that the rectangle

$$\Omega := (a', a) + i(-\gamma', b) \Subset \{\lambda \in \mathbb{C} : -\pi/8 < \arg \lambda < 7\pi/8\},$$
(1.4)

we have, uniformly on Ω ,

$$\lambda_j(\varepsilon) \to \lambda_j, \quad \varepsilon \to 0+$$

where λ_j are the resonances of P.

Notation. We use the following notation: $f = \mathcal{O}_{\ell}(g)_H$ means that $||f||_H \leq C_{\ell}g$ where the norm (or any seminorm) is in the space H, and the constant C_{ℓ} depends on ℓ . When either ℓ or H are absent then the constant is universal or the estimate is scalar, respectively. When $G = \mathcal{O}_{\ell}(g) : H_1 \to H_2$ then the operator $G : H_1 \to H_2$ has its norm bounded by $C_{\ell}g$. Also when no confusion is likely to result, we denote the operator $f \mapsto gf$ where g is a function by g.

2. MEROMORPHIC CONTINUATION

In this section we will introduce a meromorphic continuation of the weighted resolvent $\sqrt{V}R_V(\lambda)\sqrt{V}$ from Im $\lambda > 0$ to the strip Im $\lambda > -\gamma$ under the assumption (1.1). As in [10], we define the resonances of P as the poles of this meromorphic continuation, with agreement of multiplicities. For a detailed presentation, we refer to [10].

Let $R_0(\lambda) := (-\Delta - \lambda^2)^{-1}$ be the free resolvent. For Im $\lambda > 0$, the resolvent equation

$$R_0(\lambda) - R_V(\lambda) - R_V(\lambda)VR_0(\lambda) = 0$$

implies

$$(I - \sqrt{V}R_V(\lambda)\sqrt{V})(I + \sqrt{V}R_0(\lambda)\sqrt{V}) = I.$$

Since $R_0(\lambda) = \mathcal{O}(|\operatorname{Im} \lambda|^{-1}) : L^2 \to L^2$, then for $\operatorname{Im} \lambda$ large, $I + \sqrt{V}R_0(\lambda)\sqrt{V}$ is invertible by a Neumann series argument and

$$I - \sqrt{V}R_V(\lambda)\sqrt{V} = (I + \sqrt{V}R_0(\lambda)\sqrt{V})^{-1}.$$
(2.1)

We will show that the right side of (2.1) has a meromorphic continuation. For that, we recall some bounds of the free resolvent with exponential weights, see [11] for details, to prove the following lemma: **Lemma 1.** For any a > 0 and $\gamma' < \gamma$,

$$\lambda \mapsto (I + \sqrt{V}R_0(\lambda)\sqrt{V})^{-1}, \quad \operatorname{Re} \lambda > a, \ \operatorname{Im} \lambda > -\gamma',$$

is a meromorphic family of operators on $L^2(\mathbb{R}^n)$ with poles of finite rank.

Proof. Choose $\varphi \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ satisfying $\varphi(x) = |x|$ for large |x|, it is well known that for each c > 0, the weighted resolvent:

$$e^{-c\varphi}R_0(\lambda)e^{-c\varphi}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

extends analytically across $\operatorname{Re} \lambda > 0$ to the strip $\operatorname{Im} \lambda > -c$, see [11, §1] and references given there. Moreover, Gannot [11, §1] proved that for any $a, c, \varepsilon > 0$ and $\alpha \in \mathbb{N}^n$, $|\alpha| \leq 2$ there exists $C_{\alpha} = C_{\alpha}(a, c, \varepsilon)$ such that

$$\|D^{\alpha}(e^{-c\varphi}R_0(\lambda)e^{-c\varphi})\|_{L^2\to L^2} \le C_{\alpha}|\lambda|^{|\alpha|-1}, \quad \text{for Im } \lambda > -c + \varepsilon, \ \text{Re } \lambda > a.$$
(2.2)

In particular, for $\operatorname{Re} \lambda > a$ and $\operatorname{Im} \lambda > -\gamma'$,

$$\lambda \mapsto e^{-\gamma'\varphi} R_0(\lambda) e^{-\gamma'\varphi}$$

is an analytic family of operators $L^2 \to H^2$. Since $\lim_{|x|\to\infty} \sqrt{V(x)}e^{\gamma'\varphi(x)} = 0$ by (1.1), it is easy to see that $\sqrt{V}e^{\gamma'\varphi}: H^2 \to L^2$ is compact. Hence,

$$\lambda \mapsto \sqrt{V} R_0(\lambda) \sqrt{V} = \sqrt{V} e^{\gamma' \varphi} (e^{-\gamma' \varphi} R_0(\lambda) e^{-\gamma' \varphi}) \sqrt{V} e^{\gamma' \varphi}$$

is an analytic family of compact operators $L^2 \to L^2$ for $\operatorname{Re} \lambda > a$, $\operatorname{Im} \lambda > -\gamma'$. Recalling that $I + \sqrt{V}R_0(\lambda)\sqrt{V}$ is invertible for $\operatorname{Im} \lambda \gg 1$, then by the analytic Fredholm theory – see [12, §C.4], $\lambda \mapsto (I + \sqrt{V}R_0(\lambda)\sqrt{V})^{-1}$ is a meromorphic family of operators in the same range of λ .

From now on, we identify the resonances λ_j , in Ω given in (1.4), with the poles of $(I + \sqrt{V}R_0(\lambda)\sqrt{V})^{-1}$, with agreement of multiplicities. More precisely, the multiplicity of resonance λ is given by

$$m(\lambda) := \frac{1}{2\pi i} \operatorname{tr} \oint_{\lambda} (I + \sqrt{V} R_0(\zeta) \sqrt{V})^{-1} \partial_{\zeta} (\sqrt{V} R_0(\zeta) \sqrt{V}) \, d\zeta, \qquad (2.3)$$

where the integral is over a positively oriented circle enclosing λ and containing no poles other than λ .

3. RESOLVENT ESTIMATES FOR THE DAVIES HARMONIC OSCILLATOR

The operator $H_c := -\Delta + cx^2$, $-\pi < \arg c \leq 0$, was used by Davies [13] to illustrate properties of non-normal differential operators. We recall some known facts about H_c and its resolvent. As established in [13], H_c is an unbounded operator on $L^2(\mathbb{R}^n)$ with the discrete spectrum given by

$$\sigma(H_c) = \{ c^{1/2}(n+2|\alpha|) : \alpha \in \mathbb{N}_0^n \}.$$
(3.1)

In particular $\sigma(H_{-i\varepsilon}) \subset e^{-i\pi/4}[0,\infty)$, then one can study the resolvent of $H_{-i\varepsilon}$ outside $e^{-i\pi/4}[0,\infty)$. Unlike the normal operators, there does not exist any constant C such that $\|(-\Delta - i\varepsilon x^2 - z)^{-1}\|_{L^2 \to L^2} \leq C \operatorname{dist}(z, e^{-i\pi/4}[0,\infty))^{-1}$. Instead, according to Hitrik–Sjöstrand–Viola [14], [4, §3] and references given there, for $\Omega \in \{z : -\pi/2 < \arg z < 0\} \setminus e^{-i\pi/4}[0,\infty)$, there exists $C = C(\Omega)$ such that

$$\frac{1}{C}e^{\varepsilon^{-\frac{1}{2}}/C} \le \|(-\Delta - i\varepsilon x^2 - z)^{-1}\|_{L^2 \to L^2} \le Ce^{C\varepsilon^{-\frac{1}{2}}}, \quad z \in \Omega.$$
(3.2)

In this section we will show how exponential weights dramatically improve the bound (3.2) for $(-\Delta - i\varepsilon x^2 - \lambda^2)^{-1}$ in the rectangle Ω given by (1.4), which will be crucial in the proof of Theorem 1.

First, note that $-\Delta_x - i\varepsilon x^2 = \mathcal{F}^{-1}(\xi^2 + i\varepsilon \Delta_\xi)\mathcal{F}$, where \mathcal{F} denotes the Fourier transform $\mathcal{F}u(\xi) = \hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi} u(x) \, dx$. Inspired by [8] and [9], we introduce a family of spectral deformations in the Fourier space as follows.

For any fixed Ω given in (1.4), we choose $\rho \in \mathcal{C}^{\infty}([0,\infty);\mathbb{R})$ with $\rho \equiv 0$ near 0 and $\rho(t) \equiv 1$ for $t \gg 1$ such that

$$0 \le \rho'(t) < \gamma^{-1} \tan \frac{\pi}{8}, \ \forall t \ge 0; \quad \Omega \Subset \{x + iy : x > 0, \ y > -\gamma \rho(x)\},$$
(3.3)

and define the map

$$\psi: \mathbb{R}^n \to \mathbb{R}^n, \quad \psi(\xi) = |\xi|^{-1} \rho(|\xi|) \,\xi, \tag{3.4}$$

then ψ is smooth with the Jacobian:

$$D\psi(\xi) = |\xi|^{-1}\rho(|\xi|)I + (|\xi|^{-2}\rho'(|\xi|) - |\xi|^{-3}\rho(|\xi|))\xi \cdot \xi^{T}.$$
(3.5)

Let A be an orthogonal matrix with n-th column $|\xi|^{-1}\xi$, then we have

$$A^{T}D\psi(\xi) A = \operatorname{diag}[|\xi|^{-1}\rho(|\xi|), \cdots, |\xi|^{-1}\rho(|\xi|), \rho'(|\xi|)].$$
(3.6)

For $\theta \in \mathbb{R}$, we consider a family of deformations:

$$\phi_{\theta}(\xi) = \xi + \theta \psi(\xi), \qquad (3.7)$$

and the corresponding unitary operators $U_{\theta}, \ \theta \in \mathbb{R}$ defined by

$$U_{\theta}u(\xi) := (\det D\phi_{\theta}(\xi))^{\frac{1}{2}}u(\phi_{\theta}(\xi)).$$
(3.8)

Using (3.6), we can compute det $D\phi_{\theta}(\xi)$ explicitly, i.e.

$$J_{\theta}(\xi) \equiv \det D\phi_{\theta}(\xi) = \det(I + \theta D\psi(\xi)) = (1 + \theta \rho'(|\xi|)) (1 + \theta |\xi|^{-1} \rho(|\xi|))^{n-1}, \quad (3.9)$$

then by (3.3), U_{θ} is invertible as det $D\phi_{\theta}(\xi) \neq 0$ for $\theta \in \mathbb{R}$, $|\theta| < \gamma$, the inverse is given by

$$U_{\theta}^{-1}v(\xi) = (\det D\phi_{\theta}(\phi_{\theta}^{-1}(\xi)))^{-\frac{1}{2}}v(\phi_{\theta}^{-1}(\xi)).$$
(3.10)

Now we consider the deformed operators of $\xi^2 + i\varepsilon \Delta_{\xi}$:

$$Q_{\varepsilon,\theta} := U_{\theta}(\xi^2 + i\varepsilon\Delta_{\xi})U_{\theta}^{-1}$$

$$= \phi_{\theta}(\xi)^2 - i\varepsilon J_{\theta}(\xi)^{-\frac{1}{2}}D_{\xi_l}J^{lj}(\xi)J_{\theta}(\xi)J^{kj}(\xi)D_{\xi_k}J_{\theta}(\xi)^{-\frac{1}{2}}$$
(3.11)

where $D_{\xi_k} = -i\partial_{\xi_k}$, $J_{\theta}(\xi) = \det D\phi_{\theta}(\xi)$, $J^{lj}(\xi) = [D\phi_{\theta}(\xi)^{-1}]_{jl}$. To extend $Q_{\varepsilon,\theta}$ to $\theta \in \mathbb{C}$, we define

$$D_{\gamma} := \{ \theta \in \mathbb{C} : |\operatorname{Re} \theta| + |\operatorname{Im} \theta| < \gamma \}.$$
(3.12)

In view of (3.3) and (3.9), $D\phi_{\theta}^{-1}$ and det $D\phi_{\theta}$ extend analytically to $\theta \in D_{\gamma}$. Therefore, we obtain that $Q_{\varepsilon,\theta}$, given by the second equation in (3.11), extends analytically to $\theta \in D_{\gamma}$.

Then we introduce some preliminary results about the spectrum of $Q_{\varepsilon,\theta}$:

Proposition 1. There exists constant $\varepsilon_0 = \varepsilon_0(\Omega, \gamma)$ such that for all $0 < \varepsilon < \varepsilon_0$ and $\theta \in D_{\gamma}$,

$$\sigma(Q_{\varepsilon,\theta}) \cap \{z \in \mathbb{C} : |z| > 1, \pi/2 < \arg z < \pi\} = \emptyset.$$

Proof. We note that for $\theta \in D_{\gamma}$, by (3.3),

$$1 - \tan\frac{\pi}{8} < 1 - |\theta||\rho'(t)| \le |1 + \theta\rho'(t)| \le 1 + |\theta||\rho'(t)| < 1 + \tan\frac{\pi}{8}, \quad \forall t \ge 0.$$

Thus, (3.9) implies that $C^{-1} < |J_{\theta}(\xi)| < C$ for some constant C > 0. Since

$$[D\phi_{\theta}(\xi)]_{jl} = \left(1 + \theta \frac{\rho(|\xi|)}{|\xi|}\right) \delta_{jl} + \frac{\theta|\xi|\rho'(|\xi|) - \theta\rho(|\xi|)}{|\xi|^3} \xi_j \xi_l$$

by (3.5), and $\rho' \in \mathcal{C}^{\infty}_{c}((0,\infty))$, together with (3.9), we conclude that

$$J_{\theta}, J_{\theta}^{-1}, J^{lj} \in \mathcal{C}_b^{\infty}(\mathbb{R}^n), \ 1 \le j, l \le n.$$
(3.13)

Here $\mathcal{C}_b^{\infty}(\mathbb{R}^n) := \{ u \in \mathcal{C}^{\infty}(\mathbb{R}^n) : |\partial^{\alpha} u| \leq C_{\alpha} \text{ for all } \alpha \in \mathbb{N}_0^n \}.$ Hence we have

$$Q_{\varepsilon,\theta} = \phi_{\theta}(\xi)^2 - i\varepsilon J^{kj}(\xi) J^{lj}(\xi) D_{\xi_k} D_{\xi_l} + \varepsilon a_j(\xi) D_{\xi_j} + \varepsilon b(\xi), \qquad (3.14)$$

where $a_j, b \in \mathcal{C}_b^{\infty}(\mathbb{R}^n)$. Let $h = \sqrt{\varepsilon}$, then $Q_{\varepsilon,\theta} = q_{\theta}(\xi, hD_{\xi}; h)$ is a semiclassical differential operator – see Zworski [15, §4], with the symbol

$$q_{\theta}(\xi,\xi^*;h) = \phi_{\theta}(\xi)^2 - i(D\phi_{\theta}(\xi)^{-2}\xi^*) \cdot \xi^* + ha_j(\xi)\xi_j^* + h^2b(\xi), \qquad (3.15)$$

where (ξ, ξ^*) are coordinates of $T^*\mathbb{R}^n$, $D\phi_\theta(\xi)^{-2} = (D\phi_\theta(\xi)^{-1})^T (D\phi_\theta(\xi)^{-1})$ since $D\phi_\theta(\xi)$ is a symmetric matrix. Choose $m(\xi, \xi^*) = 1 + \xi^2 + \xi^{*2}$ as an order function, we recall the symbol class S(m) from [15, §4.4],

$$S(m) := \{ a \in \mathcal{C}^{\infty} : |\partial^{\alpha} a| \le C_{\alpha} m \quad \text{for } \forall \alpha \in \mathbb{N}_{0}^{2n} \}.$$
(3.16)

Then by (3.3), (3.7) and (3.13), we have $q_{\theta} \in S(m)$. Hence it suffices to show that there exists constant $h_0 > 0$ such that for $h < h_0$,

$$q_{\theta} - z$$
 is elliptic in $S(m)$ for $|z| > 1$, $\pi/2 < \arg z < \pi$.

For a detailed introduction of general elliptic theory, we refer to [15, §4].

Using (3.4) we calculate:

$$\phi_{\theta}(\xi)^{2} = (\xi + \theta\psi(\xi)) \cdot (\xi + \theta\psi(\xi)) = (|\xi| + \theta\rho(|\xi|))^{2}.$$
(3.17)

Then for $\theta \in D_{\gamma}$, by (3.3), we have

$$-\pi/4 < \arg \phi_{\theta}(\xi)^{2} < \pi/4, \quad |\phi_{\theta}(\xi)^{2}| > \left(1 - \tan \frac{\pi}{8}\right)^{2} |\xi|^{2}.$$
(3.18)

To obtain similar bounds for the argument and modulus of $(D\phi_{\theta}(\xi)^{-2}\xi^*) \cdot \xi^*$, we recall (3.6) to compute

$$(D\phi_{\theta}^{-2}\xi^*) \cdot \xi^* = (1 + \theta\rho(|\xi|)|\xi|^{-1})^{-2}(\eta_1^{*2} + \dots + \eta_{n-1}^{*2}) + (1 + \theta\rho'(|\xi|))^{-2}\eta_n^{*2}, \qquad (3.19)$$

where $\eta^* = A^T \xi^* \in \mathbb{R}^n$ with the same orthogonal matrix A as in (3.6). By (3.3), for $\theta \in D_{\gamma}$, we have

$$\pm \operatorname{Im} \theta \ge 0 \implies 0 \le \pm \arg(1 + \theta \rho(|\xi|) |\xi|^{-1}), \ \pm \arg(1 + \theta \rho'(|\xi|)) < \pi/8,$$

Hence, for all $\theta \in D_{\gamma}$,

$$\pm \operatorname{Im} \theta \ge 0 \implies 0 \le \mp \arg \left(D\phi_{\theta}^{-2} \xi^* \right) \cdot \xi^* < \pi/4, \tag{3.20}$$

and by applying the following basic inequality with (3.3) to (3.19),

$$|r_1 e^{i\theta_1} + r_2 e^{i\theta_2}|^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2) \ge \frac{1 - |\cos(\theta_1 - \theta_2)|}{2} (r_1 + r_2)^2, \quad (3.21)$$

we also obtain that for all $\theta \in D_{\gamma}$,

$$|(D\phi_{\theta}^{-2}\xi^*) \cdot \xi^*| \ge C|\eta^*|^2 = C|\xi^*|^2.$$
(3.22)

Since $\arg(\phi_{\theta}(\xi)^2 - z) \in (-\pi/2, \pi/4)$ for $\pi/2 < \arg z < \pi$ and $\arg -i(D\phi_{\theta}^{-2}\xi^*) \cdot \xi^* \in (-3\pi/4, -\pi/4)$ by (3.20), using (3.21) together with (3.18) and (3.22), we have

$$\begin{aligned} |\phi_{\theta}(\xi)^{2} - z - i(D\phi_{\theta}^{-2}\xi^{*}) \cdot \xi^{*}| &\geq C|\phi_{\theta}(\xi)^{2} - z| + C| - i(D\phi_{\theta}^{-2}\xi^{*}) \cdot \xi^{*}| \\ &\geq C|\phi_{\theta}(\xi)^{2}| + C|z| + C|\xi^{*}|^{2} \\ &\geq C(1 + |\xi|^{2} + |\xi^{*}|^{2}) = Cm. \end{aligned}$$
(3.23)

Then by (3.15), we conclude that there exists $h_0 > 0$ such that for all $h < h_0$, $|q_\theta - z| \ge Cm$, which completes the proof.

Proposition 2. For any $\beta \in (\gamma', \gamma)$ satisfying

$$\Omega \in \{x + iy : x > 0, y > -\beta\rho(x)\},\tag{3.24}$$

there exists $\varepsilon_0 = \varepsilon_0(\Omega, \gamma, \beta)$ such that for all $0 < \varepsilon < \varepsilon_0$,

$$\sigma(Q_{\varepsilon,-i\beta}) \cap \{\lambda^2 : \lambda \in \Omega\} = \emptyset.$$

Proof. As in the proof of Proposition 1, it suffices to show that there exists $h_0 = h_0(\Omega, \gamma, \beta)$ such that for $0 < h < h_0$,

 $q_{-i\beta}(\xi,\xi^*;h) - \lambda^2$ is elliptic in S(m) for $\lambda \in \Omega$.

Recalling $\arg -i(D\phi_{-i\beta}^{-2}\xi^*) \cdot \xi^* \in [-\pi/2, -\pi/4)$ by (3.20), in order to apply (3.21), we claim that

$$\exists \delta > 0 \text{ s.t. } \arg(\phi_{-i\beta}(\xi)^2 - \lambda^2) \le \pi/2 - \delta \text{ or } \ge 3\pi/4 + \delta, \text{ for all } \lambda \in \Omega, \, \xi \in \mathbb{R}^n.$$
(3.25)

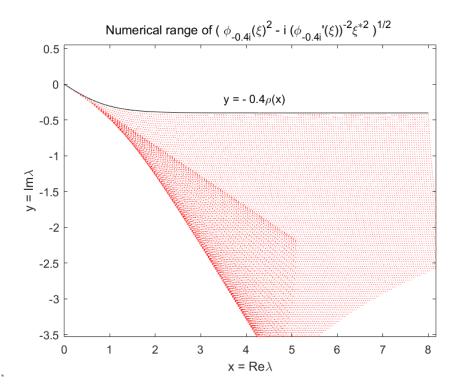


FIG. 1. An illustration of the results of Proposition 2 in the case of dim = 1, $\beta = 0.4$, which shows that the numerical range of the principal symbol of $Q_{\varepsilon,-0.4i}$ avoids the region $\{\lambda^2 : \lambda \in \Omega\}$. We choose $\rho(\cdot) = 0.4 \tanh(\cdot)$ to compute the numerical range of $(\phi_{-0.4i}(\xi)^2 - i(\phi'_{-0.4i}(\xi))^{-2}\xi^{*2})^{1/2}$.

We notice that for $|\xi| \gg 1$, $\phi_{-i\beta}(\xi)^2 = (|\xi| - i\beta)^2$ by (3.17), thus $\arg(\phi_{-i\beta}(\xi)^2 - \lambda^2) \in (-\pi/4, 0)$, in other words, there exists some large R such that (3.25) holds for $|\xi| > R$ with $\delta = \pi/2$. It remains to show that (3.25) holds for all $|\xi| \leq R$ and $\lambda \in \Omega$. We argue by contradiction: if it does not hold, there must exist $\lambda \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$ such that $\arg(\phi_{-i\beta}(\xi)^2 - \lambda^2) \in [\pi/2, 3\pi/4]$, i.e.

$$0 \le -\operatorname{Re}\left((|\xi| - i\beta\rho(|\xi|))^2 - \lambda^2\right) \le \operatorname{Im}\left((|\xi| - i\beta\rho(|\xi|))^2 - \lambda^2\right),$$

which immediately implies $\operatorname{Im} \lambda \leq 0$. Let $t = |\xi|$ and write $\lambda = x - iy$, then we have

$$x^{2} - y^{2} - t^{2} + \beta^{2} \rho(t)^{2} \le 2xy - 2\beta t \rho(t)$$
(3.26)

$$\beta t \rho(t) \le xy \tag{3.27}$$

Since x > 0 and $0 \le y < \beta \rho(x)$ by (3.24), then (3.26) implies that

$$x^{2} - 2\beta x \rho(x) - \beta^{2} \rho(x)^{2} < t^{2} - 2\beta t \rho(t) - \beta^{2} \rho(t)^{2}$$

Let $S(x) = x^2 - 2\beta x \rho(x) - \beta^2 \rho(x)^2$, by (3.3),

$$S'(x) = 2x \left(1 - \beta \frac{\rho(x)}{x} - \beta \rho'(x) - \beta \frac{\rho(x)}{x} \cdot \beta \rho'(x) \right)$$
$$> 2x \left(1 - 2 \tan \frac{\pi}{8} - \tan^2 \frac{\pi}{8} \right) = 0,$$

thus $S(x) < S(t) \implies x < t$. Recalling that ρ is non-decreasing, we have $\beta t \rho(t) \ge \beta x \rho(x) > xy$, which contradicts (3.27). Hence (3.25) holds, using (3.21) and (3.22), we obtain that

$$|\phi_{-i\beta}(\xi)^2 - \lambda^2 - i(D\phi_{-i\beta}^{-2}\xi^*) \cdot \xi^*| \ge C(\delta)(|(|\xi| - i\beta\rho(|\xi|))^2 - \lambda^2| + |\xi^*|^2).$$

Since for $|\xi| \gg 1$,

$$|(|\xi| - i\beta\rho(|\xi|))^2 - \lambda^2| = |(|\xi| - i\beta)^2 - \lambda^2| \ge |\xi|^2 - \beta^2 - |\lambda|^2,$$

there exists $R = R(\Omega, \beta) > 0$ such that $|(|\xi| - i\beta\rho(|\xi|))^2 - \lambda^2| \ge (1 + |\xi|^2)/2$ whenever $|\xi| > R$. We also note that, by (3.24),

dist
$$(\{t - i\beta\rho(t) : t \ge 0\}, \pm \Omega) \ge C = C(\Omega, \gamma, \beta) > 0,$$

thus $|(|\xi| - i\beta\rho(|\xi|))^2 - \lambda^2| \ge C^2 \ge C^2(1+R^2)^{-1}(1+|\xi|^2)$ for $|\xi| \le R$. Hence $|\phi_{-i\beta}(\xi)^2 - \lambda^2 - i(D\phi_{-i\beta}^{-2}\xi^*) \cdot \xi^*| \ge C(1+|\xi|^2+|\xi^*|^2)$, where C determined by Ω, γ, β . Then by (3.15), we conclude that there exist $h_0 = h_0(\Omega, \gamma, \beta)$ and $C = C(\Omega, \gamma, \beta) > 0$ such that

for all
$$0 < h < h_0, \ \lambda \in \Omega, \quad |q_{-i\beta}(\xi, \xi^*; h) - \lambda^2| \ge Cm(\xi, \xi^*),$$
 (3.28)

which completes the proof.

Now we state the main result of this section:

Lemma 2. For any 0 < a' < a < b and $\gamma' < \gamma$ such that the rectangle

$$\Omega := (a', a) + i(-\gamma', b) \Subset \{\lambda \in \mathbb{C} : -\pi/8 < \arg \lambda < 7\pi/8\},\$$

there exist constant $C = C(\Omega, \gamma) > 0$ and $\varepsilon_0 = \varepsilon_0(\Omega, \gamma) > 0$ such that uniformly for $0 < \varepsilon < \varepsilon_0$,

$$\|e^{-\gamma|x|}(-\Delta - i\varepsilon x^2 - \lambda^2)^{-1}e^{-\gamma|x|}\|_{L^2 \to L^2} \le C, \quad \forall \lambda \in \Omega.$$

Proof. We consider the matrix element

$$B_{f,g}^{\varepsilon}(\lambda) := \langle e^{-\gamma|x|} (-\Delta - i\varepsilon x^2 - \lambda^2)^{-1} e^{-\gamma|x|} f, g \rangle_{L^2_x}, \quad \text{for } f, g \in L^2(\mathbb{R}^n),$$

where $\langle u, v \rangle_{L^2_x} = \int_{\mathbb{R}^n} u \bar{v} \, dx$ is the standard L^2 inner product. It suffices to show that there exist C, ε_0 such that uniformly for $0 < \varepsilon < \varepsilon_0$,

$$|B_{f,g}^{\varepsilon}(\lambda)| \le C ||f||_{L^2} ||g||_{L^2}, \text{ for all } f, g \in L^2, \ \lambda \in \Omega.$$
 (3.29)

Recalling (3.1), both $-\Delta_x - i\varepsilon x^2 - \lambda^2$ and $\xi^2 + i\varepsilon \Delta_{\xi} - \lambda^2$ are invertible for $\lambda \in \Omega$. Then we have

$$B_{f,g}^{\varepsilon}(\lambda) = \langle (-\Delta_x - i\varepsilon x^2 - \lambda^2)^{-1} e^{-\gamma|x|} f, \ e^{-\gamma|x|} g \rangle_{L^2_x}$$

$$= \langle \mathcal{F}^{-1} (\xi^2 + i\varepsilon \Delta_{\xi} - \lambda^2)^{-1} \mathcal{F} e^{-\gamma|x|} f, \ e^{-\gamma|x|} g \rangle_{L^2_x}$$

$$= \langle (\xi^2 + i\varepsilon \Delta_{\xi} - \lambda^2)^{-1} \mathcal{F} (e^{-\gamma|x|} f)(\xi), \ \mathcal{F} (e^{-\gamma|x|} g)(\xi) \rangle_{L^2_{\xi}}.$$

(3.30)

Let $F_{\gamma}(\xi) := \mathcal{F}(e^{-\gamma|x|}f)(\xi)$ and $G_{\gamma}(\xi) := \mathcal{F}(e^{-\gamma|x|}g)(\xi)$, recalling the formula

$$\mathcal{F}(e^{-|x|})(\xi) = c_n(1+\xi^2)^{-\frac{n+1}{2}}, \quad c_n = (2\pi)^{\frac{n}{2}}\Gamma((n+1)/2)\pi^{-\frac{n+1}{2}},$$

then $F_{\gamma} = K_{\gamma} * \hat{f}$ and $G_{\gamma} = K_{\gamma} * \hat{g}$, where $K_{\gamma}(\xi) = c_n \gamma (\gamma^2 + \xi^2)^{-\frac{n+1}{2}}$.

First we consider, for $\theta \in \mathbb{R}$, $|\theta| < \gamma$ and U_{θ} defined by (3.8), the integral kernel of the map $U_{\theta} \circ (K_{\gamma} *)$:

$$K(\xi,\eta;\theta) := (\det D\phi_{\theta}(\xi))^{\frac{1}{2}} K_{\gamma}(\phi_{\theta}(\xi) - \eta), \quad \xi, \eta \in \mathbb{R}^{n}.$$

We claim that $K(\xi, \eta; \theta)$ has an analytic extension to $\theta \in D_{\gamma}$. Since K_{γ} extends analytically to the strip $\{\xi \in \mathbb{C}^n : |\operatorname{Im} \xi| < \gamma\}$, it suffices to show that $|\operatorname{Im}(\phi_{\theta}(\xi) - \eta)| = |\operatorname{Im} \theta \psi(\xi)| < \gamma$, which is a direct consequence of $\theta \in D_{\gamma}$ and $|\psi(\xi)| \leq 1$ by (3.4). Then for $\theta \in D_{\gamma}$, using (3.3) and (3.9), we can estimate $K(\xi, \eta; \theta)$ as follows:

$$|K(\xi,\eta;\theta)| \le C\gamma |\gamma^{2} + (\xi + \theta\psi(\xi) - \eta)^{2}|^{-\frac{n+1}{2}}$$

$$\le C\gamma |\gamma^{2} - |\operatorname{Im}\theta|^{2}|\psi(\xi)|^{2} + (\xi - \eta + \operatorname{Re}\theta\psi(\xi))^{2}|^{-\frac{n+1}{2}}$$

$$\le C\gamma (\gamma^{2} - |\operatorname{Im}\theta|^{2} + (|\xi - \eta| - |\operatorname{Re}\theta|)^{2})^{-\frac{n+1}{2}}$$

thus

$$\max\left\{\sup_{\xi\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(\xi,\eta;\theta)|d\eta,\ \sup_{\eta\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(\xi,\eta;\theta)|d\xi\right\}$$
$$\leq C\gamma\int_{x\in\mathbb{R}^n}(\gamma^2-|\operatorname{Im}\theta|^2+(|x|-|\operatorname{Re}\theta|)^2)^{-\frac{n+1}{2}}dx\leq C(\gamma,\theta).$$
(3.31)

Hence, by Schur's criterion, $U_{\theta} \circ (K_{\gamma} *)$, first defined for $\theta \in D_{\gamma} \cap \mathbb{R}$, with the integral kernel $K(\xi, \eta; \theta)$, extends to $\theta \in D_{\gamma}$ as an analytic family of operators $L^2 \to L^2$. In particular,

$$D_{\gamma} \ni \theta \mapsto U_{\theta}F_{\gamma} = U_{\theta}(K_{\gamma} * \hat{f}) \text{ and } U_{\theta}G_{\gamma} = U_{\theta}(K_{\gamma} * \hat{g}),$$

are two analytic families of functions in $L^2(\mathbb{R}^n)$.

Now we define

$$B_{f,g}^{\varepsilon}(\lambda;\theta) = \langle (Q_{\varepsilon,\theta} - \lambda^2)^{-1} U_{\theta} F_{\gamma}, U_{\bar{\theta}} G_{\gamma} \rangle$$

for $\theta \in D_{\gamma}$, with $Q_{\varepsilon,\theta}$ given by (3.11), where we write $U_{\bar{\theta}}G_{\gamma}$ instead of $U_{\theta}G_{\gamma}$. Then by Proposition 1, there exists $\varepsilon_0 = \varepsilon_0(\Omega, \gamma)$ such that for all $0 < \varepsilon < \varepsilon_0$, and $|\lambda| > 1$ with $\pi/4 < \arg \lambda < \pi/2$,

$$D_{\gamma} \ni \theta \mapsto B^{\varepsilon}_{f,q}(\lambda; \theta)$$
 is analytic

However, for $\theta \in D_{\gamma} \cap \mathbb{R}$, since U_{θ} is unitary, by (3.30) we have

$$B_{f,g}^{\varepsilon}(\lambda;\theta) = \langle U_{\theta}(\xi^{2} + i\varepsilon\Delta_{\xi} - \lambda^{2})^{-1}U_{\theta}^{-1}U_{\theta}F_{\gamma}, U_{\theta}G_{\gamma} \rangle$$
$$= \langle U_{\theta}(\xi^{2} + i\varepsilon\Delta_{\xi} - \lambda^{2})^{-1}F_{\gamma}, U_{\theta}G_{\gamma} \rangle$$
$$= \langle (\xi^{2} + i\varepsilon\Delta_{\xi} - \lambda^{2})^{-1}F_{\gamma}, G_{\gamma} \rangle = B_{f,g}^{\varepsilon}(\lambda).$$

Thus by analyticity, $B_{f,g}^{\varepsilon}(\lambda;\theta) \equiv B_{f,g}^{\varepsilon}(\lambda), \ \forall \theta \in D_{\gamma}$ whenever $|\lambda| > 1, \ \pi/4 < \arg \lambda < \pi/2$. In particular, for fixed $\beta \in (\gamma', \gamma)$ satisfying (3.24),

$$B_{f,g}^{\varepsilon}(\lambda) = B_{f,g}^{\varepsilon}(\lambda; -i\beta)$$
 whenever $|\lambda| > 1, \pi/4 < \arg \lambda < \pi/2.$

In view of Proposition 2 and (3.1), both $B_{f,g}^{\varepsilon}(\lambda)$ and $B_{f,g}^{\varepsilon}(\lambda; -i\beta)$ are analytic in Ω . Without loss of generality, we may assume that a > 1 in (1.4), then

$$\Omega \cap \{\lambda : |\lambda| > 1, \pi/4 < \arg \lambda < \pi/2\} \neq \emptyset,$$

where $B_{f,g}^{\varepsilon}(\lambda)$ and $B_{f,g}^{\varepsilon}(\lambda; -i\beta)$ coincide. Hence by analyticity, we conclude that for each $0 < \varepsilon < \varepsilon_0$,

$$B_{f,g}^{\varepsilon}(\lambda) = B_{f,g}^{\varepsilon}(\lambda; -i\beta) = \langle (Q_{\varepsilon, -i\beta} - \lambda^2)^{-1} U_{-i\beta} F_{\gamma}, U_{i\beta} G_{\gamma} \rangle, \quad \forall \lambda \in \Omega.$$
(3.32)

By the elliptic theory of semiclassical differential operators – see [15, §4.7], (3.28) implies that there exists $\varepsilon_0 = \varepsilon_0(\Omega, \gamma, \beta)$ such that for all $0 < \varepsilon < \varepsilon_0$,

$$\|(Q_{\varepsilon,-i\beta} - \lambda^2)^{-1}\|_{L^2 \to L^2} \le C(\Omega,\gamma,\beta), \quad \forall \lambda \in \Omega.$$
(3.33)

Recalling (3.31), by Schur's criterion, we obtain that

$$\begin{aligned} \|U_{-i\beta}F_{\gamma}\|_{L^{2}} &= \|U_{-i\beta}\circ(K_{\gamma}*\hat{f})\|_{L^{2}} \le C(\gamma,\beta)\|\hat{f}\|_{L^{2}} = C(\gamma,\beta)\|f\|_{L^{2}} \\ \|U_{i\beta}G_{\gamma}\|_{L^{2}} &= \|U_{i\beta}\circ(K_{\gamma}*\hat{g})\|_{L^{2}} \le C(\gamma,\beta)\|\hat{g}\|_{L^{2}} = C(\gamma,\beta)\|g\|_{L^{2}} \end{aligned}$$
(3.34)

Combining (3.32), (3.33) and (3.34), also noticing that β can be determined by Ω, γ , we obtain (3.29) with $C = C(\Omega, \gamma)$, which completes the proof.

4. EIGENVALUES OF THE REGULARIZED OPERATOR

In this section we will review the meromorphy of the resolvent

$$R_{V,\varepsilon}(\lambda) := (P_{\varepsilon} - \lambda^2)^{-1}, \quad \varepsilon > 0,$$

with P_{ε} in (1.2), in a similar form to the meromorphic continuation of the weighted resolvent $\sqrt{V}R_V(\lambda)\sqrt{V}$ given by (2.1).

First we write $R_{\varepsilon}(\lambda) := (-\Delta - i\varepsilon x^2 - \lambda^2)^{-1}$ and recall

$$R_{\varepsilon}(\lambda) = \mathcal{O}_{\delta}(1/|\lambda|) : L^2 \to L^2, \quad \delta < \arg \lambda < 3\pi/4 - \delta, \ |\lambda| > \delta, \tag{4.1}$$

which follows from (semiclassical) ellipticity. Then

$$(P_{\varepsilon} - \lambda^2)R_{\varepsilon}(\lambda) = I + VR_{\varepsilon}(\lambda), \quad -\pi/8 < \arg \lambda < 7\pi/8.$$
(4.2)

In view of (4.1), $I + VR_{\varepsilon}(\lambda)$ is invertible for $\pi/4 < \arg \lambda < \pi/2$, $|\lambda| \gg 1$. Since $R_{\varepsilon}(\lambda) : L^2 \to H^2$ is analytic in $\{\lambda : -\pi/8 < \arg \lambda < 7\pi/8\}$, see (3.1), $V : H^2 \to L^2$ is compact by (1.1), we have $\lambda \mapsto VR_{\varepsilon}(\lambda)$ is an analytic family of compact operators for $-\pi/8 < \arg \lambda < 7\pi/8$. Hence $\lambda \mapsto (I + VR_{\varepsilon}(\lambda))^{-1}$ is a meromorphic family of operators in the same range of λ . Using (4.2), we conclude that $R_{V,\varepsilon}(\lambda) = R_{\varepsilon}(\lambda)(I + VR_{\varepsilon}(\lambda))^{-1}$ is meromorphic for $-\pi/8 < \arg \lambda < 7\pi/8$ (in fact $R_{V,\varepsilon}(\lambda)$ is meromorphic for $\lambda \in \mathbb{C}$ by the Gohberg–Sigal factorization theorem - see [12, §C.4]), with poles $\{\lambda_j(\varepsilon)\}_{j=1}^{\infty}$, i.e. $\{\lambda_j(\varepsilon)^2\}_{j=1}^{\infty}$ are the eigenvalues of P_{ε} in $\{z \in \mathbb{C} : \arg z \neq -\pi/4\}$. Then we have

Lemma 3. For each $\varepsilon > 0$,

$$\lambda \mapsto (I + \sqrt{V}R_{\varepsilon}(\lambda)\sqrt{V})^{-1}, \quad -\pi/8 < \arg \lambda < 7\pi/8,$$

is a meromorphic family of operators on $L^2(\mathbb{R}^n)$ with poles of finite rank. Moreover,

$$m_{\varepsilon}(\lambda) := \frac{1}{2\pi i} \operatorname{tr} \oint_{\lambda} (I + \sqrt{V} R_{\varepsilon}(\zeta) \sqrt{V})^{-1} \partial_{\zeta} (\sqrt{V} R_{\varepsilon}(\zeta) \sqrt{V}) \, d\zeta, \qquad (4.3)$$

where the integral is over a positively oriented circle enclosing λ and containing no poles other than possibly λ , satisfies

$$m_{\varepsilon}(\lambda) = \frac{1}{2\pi i} \operatorname{tr} \oint_{\lambda} (\zeta^2 - P_{\varepsilon})^{-1} 2\zeta \, d\zeta.$$
(4.4)

Remark. The multiplicity of an eigenvalue λ^2 of P_{ε} can be defined by the right side of (4.4), thus Lemma 3 implies that the poles of $(I + \sqrt{V}R_{\varepsilon}(\lambda)\sqrt{V})^{-1}$ coincide with $\{\lambda_j(\varepsilon)\}_{j=1}^{\infty}$ given in (1.3), with agreement of multiplicities.

Proof. Following the above argument, it easy to see that $\lambda \mapsto \sqrt{V}R_{\varepsilon}(\lambda)\sqrt{V}$ is an analytic family of compact operators for $-\pi/8 < \arg \lambda < 7\pi/8$. Then

$$\lambda \mapsto (I + \sqrt{V}R_{\varepsilon}(\lambda)\sqrt{V})^{-1}, \quad -\pi/8 < \arg \lambda < 7\pi/8,$$

is a meromorphic family of operators, since $I + \sqrt{V}R_{\varepsilon}(\lambda)\sqrt{V}$ is invertible for $\pi/4 < \arg \lambda < \pi/2$, $|\lambda| \gg 1$ by (4.1). In this range of λ , $I + VR_{\varepsilon}(\lambda)$ is also invertible by the Neumann series argument, thus we have

$$(P_{\varepsilon} - \lambda^{2})^{-1} = R_{\varepsilon}(\lambda)(I + VR_{\varepsilon}(\lambda))^{-1}$$

$$= R_{\varepsilon}(\lambda)\sum_{j=0}^{\infty} (-1)^{j}(VR_{\varepsilon}(\lambda))^{j}$$

$$= R_{\varepsilon}(\lambda)(I - \sqrt{V}\sum_{j=0}^{\infty} (-1)^{j}(\sqrt{V}R_{\varepsilon}(\lambda)\sqrt{V})^{j}\sqrt{V}R_{\varepsilon}(\lambda))$$

$$= R_{\varepsilon}(\lambda)[I - \sqrt{V}(I + \sqrt{V}R_{\varepsilon}(\lambda)\sqrt{V})^{-1}\sqrt{V}R_{\varepsilon}(\lambda)].$$
(4.5)

Since both sides of (4.5) are meromorphic for $-\pi/8 < \arg \lambda < 7\pi/8$, by meromorphy, we conclude that (4.5) holds for all $-\pi/8 < \arg \lambda < 7\pi/8$, as an identity between meromorphic families of operators.

To obtain the multiplicity formula, we fix any λ with $-\pi/8 < \arg \lambda < 7\pi/8$, then there exists a neighborhood $\lambda \in U$ in this half plane and finite rank operators A_j , $1 \leq j \leq J$ such that $(I + \sqrt{V}R_{\varepsilon}(\zeta)\sqrt{V})^{-1} - \sum_{j=1}^{J} \frac{A_j}{(\zeta-\lambda)^j}$ is analytic in $\zeta \in U$. Let $\mathcal{C}_{\lambda} \subset U$ be a positively oriented circle enclosing λ and containing no poles of $(I + \sqrt{V}R_{\varepsilon}(\zeta)\sqrt{V})^{-1}$ other than possibly λ , thus it also contains no poles of $(\zeta^2 - P_{\varepsilon})^{-1}$ other than possibly λ as a consequence of (4.5). On the one hand, we can compute

$$m_{\varepsilon}(\lambda) = \frac{1}{2\pi i} \operatorname{tr} \int_{\mathcal{C}_{\lambda}} (I + \sqrt{V} R_{\varepsilon}(\zeta) \sqrt{V})^{-1} \sqrt{V} R_{\varepsilon}(\zeta)^{2} \sqrt{V} 2\zeta d\zeta$$

$$= \frac{1}{2\pi i} \operatorname{tr} \int_{\mathcal{C}_{\lambda}} \sum_{j=1}^{J} \frac{A_{j} \sqrt{V} R_{\varepsilon}(\zeta)^{2} 2\zeta \sqrt{V}}{(\zeta - \lambda)^{j}} d\zeta$$

$$= \sum_{j=1}^{J} \sum_{k=0}^{j-1} \frac{1}{k! (j - 1 - k)!} \operatorname{tr} A_{j} \sqrt{V} \partial_{\zeta}^{k} R_{\varepsilon}(\zeta) \partial_{\zeta}^{j-1-k} (R_{\varepsilon}(\zeta) 2\zeta) \sqrt{V}.$$

(4.6)

On the other hand, by (4.5), we have

$$\frac{1}{2\pi i} \operatorname{tr} \oint_{\lambda} (\zeta^{2} - P_{\varepsilon}))^{-1} 2\zeta d\zeta$$

$$= \frac{1}{2\pi i} \operatorname{tr} \int_{\mathcal{C}_{\lambda}} \sum_{j=1}^{J} \frac{R_{\varepsilon}(\zeta) 2\zeta \sqrt{V} A_{j} \sqrt{V} R_{\varepsilon}(\zeta)}{(\zeta - \lambda)^{j}} d\zeta$$

$$= \sum_{j=1}^{J} \sum_{k=0}^{j-1} \frac{1}{k! (j-1-k)!} \operatorname{tr} \partial_{\zeta}^{j-1-k} (R_{\varepsilon}(\zeta) 2\zeta) \sqrt{V} A_{j} \sqrt{V} \partial_{\zeta}^{k} R_{\varepsilon}(\zeta).$$
(4.7)

Now we compare (4.6) and (4.7), since each A_j has finite rank, we can apply cyclicity of the trace to obtain the multiplicity formula (4.4).

5. PROOF OF CONVERGENCE

The proof of convergence is based on Lemma 1, Lemma 3, with an application of the Gohberg–Sigal–Rouché theorem, see Gohberg–Sigal [16] and [12, Appendix C.].

We now state a more precise version of Theorem 1 involving the multiplicities given in (2.3) and (4.3) as follows:

Theorem 2. For any Ω given in (1.4), there exists $\delta_0 = \delta_0(\Omega)$ satisfying the following: for any $0 < \delta < \delta_0$, there exists $\varepsilon_{\delta} > 0$ such that for any $\lambda \in \Omega$ with $m(\lambda) > 0$,

$$\# \{\lambda_j(\varepsilon)\}_{j=1}^{\infty} \cap B(\lambda, \delta) = m(\lambda), \quad \text{for all } 0 < \varepsilon < \varepsilon_{\delta},$$

where $\{\lambda_j(\varepsilon)\}_{j=1}^{\infty}$ given in (1.3) is counted with multiplicity, $B(\lambda, \delta) := \{z \in \mathbb{C} : |z - \lambda| < \delta\}.$

Proof. In view of Lemma 1, the poles of $(I + \sqrt{V}R_0(\lambda)\sqrt{V})^{-1}$ are isolated in the region $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0, \operatorname{Im} \lambda > -\gamma\}$, thus there are finitely many $\lambda \in \Omega$ with $m(\lambda) > 0$, denoted by $\lambda_1, \ldots, \lambda_J$. We choose $\delta_0 > 0$ such that $B(\lambda_j, \delta_0), j = 1, \ldots, J$ are disjoint discs in Ω , then for any fixed $0 < \delta < \delta_0$ and each $\lambda \in \Omega$ with $m(\lambda) > 0$, we have

$$\|(I + \sqrt{V}R_0(\zeta)\sqrt{V})^{-1}\|_{L^2 \to L^2} < C(\delta), \quad \forall \, \zeta \in \partial B(\lambda, \delta),$$

for some constant $C(\delta) > 0$.

In order to apply the Gohberg–Sigal–Rouché theorem, we need to estimate :

$$\|I + \sqrt{V}R_{\varepsilon}(\zeta)\sqrt{V} - (I + \sqrt{V}R_0(\zeta)\sqrt{V})\|_{L^2 \to L^2}, \quad \text{for any } \zeta \in \Omega.$$

1. Choose $\chi \in \mathcal{C}^{\infty}_{c}(\mathbb{R}^{n})$ satisfying $\chi \equiv 1$ in $B_{\mathbb{R}^{n}}(0,1)$ and $\operatorname{supp} \chi \subset B_{\mathbb{R}^{n}}(0,2)$, here $B_{\mathbb{R}^{n}}(0,r) := \{x \in \mathbb{R}^{n} : |x| < r\}$, we define $\chi_{R}(x) = \chi(R^{-1}x)$ and calculate:

$$I + \sqrt{V}R_{\varepsilon}(\zeta)\sqrt{V} - (I + \sqrt{V}R_{0}(\zeta)\sqrt{V})$$

= $\sqrt{V}R_{\varepsilon}(\zeta)\sqrt{V} - \chi_{R}\sqrt{V}R_{\varepsilon}(\zeta)\chi_{R}\sqrt{V} + \sqrt{V}\chi_{R}(R_{\varepsilon}(\zeta) - R_{0}(\zeta))\chi_{R}\sqrt{V}$ (5.1)
 $- (\sqrt{V}R_{0}(\zeta)\sqrt{V} - \chi_{R}\sqrt{V}R_{0}(\zeta)\chi_{R}\sqrt{V}).$

2. The first term can be written as $(1 - \chi_R)\sqrt{V}R_{\varepsilon}(\zeta)\sqrt{V} + \chi_R\sqrt{V}R_{\varepsilon}(\zeta)(1 - \chi_R)\sqrt{V}$. Let $\tilde{\gamma} = (\gamma + \gamma')/2$, then

$$(1-\chi_R)\sqrt{V}R_{\varepsilon}(\zeta)\sqrt{V} = (1-\chi_R)\sqrt{V}e^{\tilde{\gamma}|x|}(e^{-\tilde{\gamma}|x|}R_{\varepsilon}(\zeta))e^{-\tilde{\gamma}|x|})\sqrt{V}e^{\tilde{\gamma}|x|},$$

where $|\sqrt{V(x)}e^{\tilde{\gamma}|x|}| \leq Ce^{(\tilde{\gamma}-\gamma)|x|} = Ce^{-(\gamma-\gamma')|x|/2}$. By Lemma 2, there exists $\varepsilon_0 = \varepsilon_0(\Omega, \tilde{\gamma})$ such that for any $0 < \varepsilon < \varepsilon_0$, $||e^{-\tilde{\gamma}|x|}R_{\varepsilon}(\zeta)|e^{-\tilde{\gamma}|x|}||_{L^2 \to L^2} \leq C(\Omega, \tilde{\gamma})$. Thus,

$$\|(1-\chi_R)\sqrt{V}R_{\varepsilon}(\zeta)\sqrt{V}\|_{L^2\to L^2} \le C(\Omega,\gamma)e^{-(\gamma-\gamma')R/2}, \quad \text{for any } 0 < \varepsilon < \varepsilon_0.$$

Similarly, we can bound $\|\chi_R \sqrt{V} R_{\varepsilon}(\zeta)(1-\chi_R) \sqrt{V}\|_{L^2 \to L^2}$ by the right side above. Hence for any $0 < \varepsilon < \varepsilon_0$,

$$\|\sqrt{V}R_{\varepsilon}(\zeta)\sqrt{V} - \chi_R\sqrt{V}R_{\varepsilon}(\zeta)\chi_R\sqrt{V}\|_{L^2 \to L^2} \le Ce^{-(\gamma - \gamma')R/2}, \quad \forall \zeta \in \Omega.$$
(5.2)

3. We can estimate the third term in (5.1) by a similar argument. (2.2) implies that

$$\|e^{-\tilde{\gamma}|x|}R_0(\zeta)e^{-\tilde{\gamma}|x|}\|_{L^2\to L^2} \le C(\Omega,\gamma), \quad \forall \zeta \in \Omega.$$

Hence, arguing as above, we obtain that

$$\|\sqrt{V}R_0(\zeta)\sqrt{V} - \chi_R\sqrt{V}R_0(\zeta)\chi_R\sqrt{V}\|_{L^2 \to L^2} \le Ce^{-(\gamma - \gamma')R/2}, \quad \forall \zeta \in \Omega.$$
(5.3)

4. We note that

$$\chi_R(R_\varepsilon(\zeta) - R_0(\zeta))\chi_R = i\varepsilon\,\chi_R(-\Delta - i\varepsilon x^2 - \zeta^2)^{-1}x^2(\Delta - \zeta^2)^{-1}\chi_R,$$

and recall [4] that there exists $C = C(\Omega, \chi_R)$ (independent of ε) such that

$$\|\chi_R(-\Delta - i\varepsilon x^2 - \zeta^2)^{-1} x^2 (\Delta - \zeta^2)^{-1} \chi_R\|_{L^2 \to L^2} \le C, \quad \forall \zeta \in \Omega, \ \varepsilon > 0,$$

which is proved using the method of complex scaling, see [4, §5] for details. Hence

$$\|\sqrt{V}\chi_R(R_{\varepsilon}(\zeta) - R_0(\zeta))\chi_R\sqrt{V}\|_{L^2 \to L^2} \le C(\Omega, \chi_R)\varepsilon, \quad \forall \zeta \in \Omega, \, \varepsilon > 0.$$
(5.4)

By (5.2) and (5.3), we can first fix R sufficiently large such that

$$\|\sqrt{V}R_{\varepsilon}(\zeta)\sqrt{V} - \chi_R\sqrt{V}R_{\varepsilon}(\zeta)\chi_R\sqrt{V}\|_{L^2 \to L^2} \le 1/(3C(\delta)), \quad \forall \zeta \in \Omega, \ 0 \le \varepsilon < \varepsilon_0$$

Then by (5.4), there exists $\varepsilon_{\delta} > 0$ such that for all $0 < \varepsilon < \varepsilon_{\delta}$,

$$\|\sqrt{V}\chi_R(R_{\varepsilon}(\zeta) - R_0(\zeta))\chi_R\sqrt{V}\|_{L^2 \to L^2} \le 1/(3C(\delta)), \quad \forall \zeta \in \Omega.$$

We may assume that $\varepsilon_{\delta} < \varepsilon_0$, thus by (5.1), we conclude that for each $0 < \varepsilon < \varepsilon_{\delta}$,

$$\|(I + \sqrt{V}R_0(\zeta)\sqrt{V})^{-1}(I + \sqrt{V}R_{\varepsilon}(\zeta)\sqrt{V} - (I + \sqrt{V}R_0(\zeta)\sqrt{V}))\|_{L^2 \to L^2} < 1,$$

on $\partial B(\lambda, \delta)$.

Now we apply the Gohberg–Sigal–Rouché theorem to obtain that

$$m(\lambda) = \frac{1}{2\pi i} \operatorname{tr} \int_{\partial B(\lambda,\delta)} (I + \sqrt{V} R_0(\zeta) \sqrt{V})^{-1} \partial_{\zeta} (\sqrt{V} R_0(\zeta) \sqrt{V}) \, d\zeta$$
$$= \frac{1}{2\pi i} \operatorname{tr} \int_{\partial B(\lambda,\delta)} (I + \sqrt{V} R_{\varepsilon}(\zeta) \sqrt{V})^{-1} \partial_{\zeta} (\sqrt{V} R_{\varepsilon}(\zeta) \sqrt{V}) \, d\zeta,$$

for each $0 < \varepsilon < \varepsilon_{\delta}$. Let $\lambda_1(\varepsilon), \ldots, \lambda_K(\varepsilon)$ be the distinct poles of $(I + \sqrt{V}R_{\varepsilon}(\zeta)\sqrt{V})^{-1}$ in $B(\lambda, \delta)$, then

$$m(\lambda) = \sum_{k=1}^{K} \frac{1}{2\pi i} \operatorname{tr} \oint_{\lambda_k(\varepsilon)} (I + \sqrt{V} R_{\varepsilon}(\zeta) \sqrt{V})^{-1} \partial_{\zeta} (\sqrt{V} R_{\varepsilon}(\zeta) \sqrt{V}) \, d\zeta = \sum_{k=1}^{K} m_{\varepsilon}(\lambda_k(\varepsilon)),$$

Therefore, with Lemma 3 and (4.4), we obtain that

$$\# \{\lambda_j(\varepsilon)\}_{j=1}^{\infty} \cap B(\lambda, \delta) = m(\lambda), \quad \forall \, 0 < \varepsilon < \varepsilon_{\delta},$$

which completes the proof.

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DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are available within the article.

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