RESONANCES AS VISCOSITY LIMITS FOR EXTERIOR DILATION ANALYTIC POTENTIALS

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ABSTRACT. For exterior dilation analytic potential, V, we use the method of complex scaling to show that the resonances of $-\Delta + V$, in a conic neighbourhood of the real axis, are limits of eigenvalues of $-\Delta + V - i\varepsilon x^2$ as $\varepsilon \to 0+$, if V can be analytically extended from $\mathbb{R}^n$ to a truncated cone in $\mathbb{C}^n$.

1. INTRODUCTION AND STATEMENT OF RESULTS

We extend the results of [Z2], when $V \in L^\infty_{\rm comp}$, to the case of exterior dilation analytic potentials. For motivation and pointers to the literature we refer to [Z2].

Thus, we consider

$$H := -\Delta + V,$$

where V is a real-valued potential which can be analytically extended from $\{x \in \mathbb{R}^n : |x| > R\}$, for some $R > 0$, to a truncated cone

$$\mathcal{C}^R_{\beta_0} := \{z \in \mathbb{C}^n \,:\, |\operatorname{Im} z| < \tan\beta_0 |\operatorname{Re} z| \text{ and } |\operatorname{Re} z| > R\}, \quad \beta_0 \le \pi/8.$$

We still denote the analytic extension by V and assume that

$$\lim_{\mathcal{C}^R_{\beta_0} \ni |z| \to \infty} V(z) = 0. \tag{1.1}$$

The resonances of H are defined by the Aguiliar-Balslev-Combes-Simon theory, see [HS, §16, §18], [DyZ2, §4.5] and a review in §3.

We now introduce a *regularized* operator,

$$H_\varepsilon := -\Delta - i\varepsilon x^2 + V, \quad \varepsilon > 0. \tag{1.2}$$

(We write $x^2 := x_1^2 + \cdots + x_n^2$.) It is easy to see, with details reviewed in §4, that H_ε is a non-normal unbounded operator on $L^2(\mathbb{R}^n)$ with a discrete spectrum. We have

Theorem 1. *Suppose that $\{z_j(\varepsilon)\}_{j=1}^\infty$ are the eigenvalues of H_ε. Then, uniformly on any compact subsets of $\{z \,:\, -2\beta_0 < \arg z < 3\pi/2 + 2\beta_0\}$,*

$$z_j(\varepsilon) \to z_j, \quad \varepsilon \to 0+,$$

where z_j are the resonances of H.

Notation. We use the following notation: $f = \mathcal{O}_\ell(g)_H$ means that $\|f\|_H \le C_\ell g$ where the norm (or any seminorm) is in the space H, and the constant C_ℓ depends on ℓ. When either ℓ or H are absent then the constant is universal or the estimate is scalar, respectively. When $G = \mathcal{O}_\ell(g) : H_1 \to H_2$ then the operator $G : H_1 \to H_2$ has its norm bounded by $C_\ell g$. Also when no confusion is likely to result, we denote the operator $f \mapsto gf$ where g is a function by g.

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2. SPECTRAL DEFORMATION AND ANALYTIC VECTORS

We will review several basic concepts in the Aguilar-Balslev-Combes-Simon theory, such as spectral deformation and analytic vectors. For a detailed introduction, we refer to [HS, §17] and the references given there.

Let $h \in \mathcal{C}^\infty(\mathbb{R})$ be a non-decreasing function which satisfies

$$\begin{cases} h(t) = 0, & t < 2R, \\ h(t) = 1, & t > 8R. \end{cases} \tag{2.1}$$

Moreover, we assume that

$$\sup_{t\in\mathbb{R}} h(t) + th'(t) \le 3/2. \tag{2.2}$$

We define $g : \mathbb{R}^n \to \mathbb{R}^n$ as a smooth mapping by

$$g(x) := h(|x|)\, x = \begin{cases} 0, & |x| < 2R, \\ x, & |x| > 8R, \end{cases} \tag{2.3}$$

and consider, for $\theta \in \mathbb{R}$, the related family of maps $\phi_\theta : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\phi_\theta(x) = x + \theta g(x) \tag{2.4}$$

We let Df denote the derivative of a map $f : \mathbb{R}^n \to \mathbb{R}^n$, then

$$Dg(x) = h(|x|)I + |x|^{-1}h'(|x|)x \cdot x^T.$$

Using diagonalization, It is easy to see that

$$0 \le h(|x|)I \le Dg(x) \le (h(|x|) + |x|h'(|x|))I \le 3/2\, I \tag{2.5}$$

where $A \le B$ means $B - A$ is positive semi-definite and the last inequality is implied by (2.2). Hence $\sup_{x\in\mathbb{R}^n} \|Dg(x)\| \le 3/2$, where $\|\cdot\|$ denotes the operator norm on the set of linear transformation on $\mathbb{R}^n$. We note that $D\phi_\theta(x) = I + \theta(Dg)(x)$, if $|\theta| < 2/3$, then $D\phi_\theta$ is invertible by a Neumann series argument,

$$(D\phi_\theta)^{-1} = \sum_{j=0}^{\infty}(-1)^j\theta^j(Dg)^j.$$

Hence ϕ_θ is a diffeomorphism of $\mathbb{R}^n$ for $|\theta| < 2/3$ by the inverse function theorem.

We should remark that all the above argument is valid when we extend the definition (2.4) of ϕ_θ to $\theta \in \mathbb{C}$. We have $\phi_\theta : \mathbb{R}^n \to \mathbb{C}^n$ is a diffeomorphism provided $|\theta| < 2/3$.

We now introduce the behavior of functions under the action of the maps ϕ_θ. We first define U_θ for $\theta \in \mathbb{R}$ by

$$(U_\theta f)(x) = J_\theta(x)^{1/2} f(\phi_\theta(x)) \tag{2.6}$$

where $J_\theta(x)$ is the Jacobian of ϕ_θ,

$$J_\theta(x) = \det D\phi_\theta(x) = \det(I + \theta(Dg)(x)). \tag{2.7}$$

It is east to see that $U_\theta,\ \theta \in \mathbb{R}$ is unitary on $L^2(\mathbb{R}^n)$ with the inverse U_θ^{-1} given by

$$(U_\theta^{-1} f)(x) = J_\theta(\phi_\theta^{-1}(x))^{-1/2} f(\phi_\theta^{-1}(x)). \tag{2.8}$$

(2.5) and (2.7) show that $J_\theta(x)^{1/2}$ extends analytically to complex θ provided $\theta < 2/3$. Hence, to extend the operators U_θ from $\theta \in \mathbb{R}$ to $\theta \in \mathbb{C}$, at least for small $|\theta|$, we need to find a dense set of functions f in $L^2(\mathbb{R}^n)$ that can be analytically extended on a small complex neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ such that $f \circ \phi_\theta \in L^2(\mathbb{R}^n)$. For that we introduce the set of analytic vectors in $L^2(\mathbb{R}^n)$.

Definition 1. *Let $\mathcal{A}$ be the linear space of all entire functions $f(z)$ having the property that in any conical region $\mathcal{C}_\varepsilon$,*

$$\mathcal{C}_\varepsilon := \{z \in \mathbb{C}^n \,:\, |\operatorname{Im} z| \le (1-\varepsilon)\operatorname{Re} z\},$$

for any $\varepsilon > 0$, we have for any $k \in \mathbb{N}$,

$$\lim_{z \in \mathcal{C}_\varepsilon \to \infty} |z|^k |f(z)| = 0.$$

The set of analytic vectors in $L^2(\mathbb{R}^n)$ are the restrictions to $\mathbb{R}^n$ of $\mathcal{A}$, which is also denoted by $\mathcal{A}$.

We define a domain D_{β_0} in $\mathbb{C}$ by

$$D_{\beta_0} = \{\theta \in \mathbb{C} \,:\, |\operatorname{Re}\theta| + |\operatorname{Im}\theta| < \tan\beta_0\}. \tag{2.9}$$

Note that $D_{\beta_0} \subset \{z \in \mathbb{C} \,:\, |z| < 1/2\}$ since $\beta_0 \le \pi/8$, (2.5) and (2.7) guarantee that the Jacobian J_θ is uniformly bounded for $\theta \in D_{\beta_0}$. Then, we recall the following results in [HS, Proposition 17.10]:

Proposition 2. *Let $\mathcal{U} \equiv \{U_\theta : \theta \in D_{\beta_0}\}$ be a spectral deformation family associated with vector field g defined by (2.3). Then,*

- *the map $(\theta, f) \in D_{\beta_0} \times \mathcal{A} \to U_\theta f$ is an L^2-analytic map ;*
- *for any $\theta \in D_{\beta_0}$, $U_\theta \mathcal{A}$ is dense in $L^2(\mathbb{R}^n)$.*

We conclude this section with some properties about the deformation of $\mathbb{R}^n \subset \mathbb{C}^n$ under the map ϕ_θ provided $\theta \in D_{\beta_0}$. We recall that $\phi_\theta : \mathbb{R}^n \to \mathbb{C}^n$ is injective with the Jacobian $J_\theta \neq 0$ provided $|\theta| < 2/3$. Hence $\phi_\theta(\mathbb{R}^n) \subset \mathbb{C}^n$ is an n-dimensional totally real submanifolds, see [DyZ2, §4.5]. Let $\Gamma_{a(\theta)} = \phi_\theta(\mathbb{R}^n)$, where $a(\theta) \in (-\pi/2, \pi/2)$ is defined by

$$a(\theta) = \arg(1+\theta). \tag{2.10}$$

In the literature about complex scaling, one can define $L^2(\Gamma_{a(\theta)})$ with volume element $|dw| = |J_\theta(x)|\, dx$ where $w = \phi_\theta(x)$ are the coordinates on $\Gamma_{a(\theta)}$, see [DyZ2, §2.7, §4.5] for details. Then we have the following:

Proposition 3. *For any $\theta \in D_{\beta_0}$, $\Gamma_{a(\theta)}$ satisfies*

$$\begin{gathered} \Gamma_{a(\theta)} \cap B_{\mathbb{C}^n}(0, 2R) = B_{\mathbb{R}^n}(0, 2R), \\ \Gamma_{a(\theta)} \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, 12R) = e^{ia(\theta)}\mathbb{R}^n \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, 12R), \\ \Gamma_{a(\theta)} \subset \mathbb{R}^n \cup \mathcal{C}^R_{\beta_0}. \end{gathered} \tag{2.11}$$

Furthermore, the spectral deformation operator U_θ extends to an isometry:

$$U_\theta : L^2(\Gamma_{a(\theta)}) \to L^2(\mathbb{R}^n).$$

Proof. In view of (2.3) and (2.4), it is easy to see that $\Gamma_{a(\theta)} = \phi_\theta(\mathbb{R}^n)$ satisfies the first two equations of (2.11). For $\theta \in D_{\beta_0}$, we have

$$\frac{|\operatorname{Im}\phi_\theta(x)|}{|\operatorname{Re}\phi_\theta(x)|} = \frac{|\operatorname{Im}\theta||\chi(|x|)|}{|1+\operatorname{Re}\theta\chi(|x|)|} \leq \frac{|\operatorname{Im}\theta|}{1-|\operatorname{Re}\theta|} < \tan\beta_0,$$

where the last inequality is implied by (2.9). Moreover, $\phi_\theta(x) = x$ for $|x| < 2R$, and $|\operatorname{Re}\phi_\theta(x)| \geq (1 - |\operatorname{Re}\theta|)|x| > (1-\tan\beta_0)|x| > |x|/2 \geq R$ provided $|x| \geq 2R$, since $\beta_0 \leq \pi/8$. Hence $\Gamma_{a(\theta)} \subset \mathbb{R}^n \cup \mathcal{C}^R_{\beta_0}$.

Now we assume that $\theta \in D_{\beta_0}$, for any $f \in L^2(\Gamma_{a(\theta)})$, we can define $U_\theta f$ on $\mathbb{R}^n$ by (2.6). To see $U_\theta f \in L^2(\mathbb{R}^n)$, we compute directly:

$$\begin{aligned} \int_{\mathbb{R}^n} |U_\theta f(x)|^2\, dx &= \int_{\mathbb{R}^n} |J_\theta(x)^{1/2} f(\phi_\theta(x))|^2\, dx \\ &= \int_{\mathbb{R}^n} |f(\phi_\theta(x))|^2 |J_\theta(x)|\, dx \\ &= \int_{\Gamma_{a(\theta)}} |f(w)|^2\, |dw| < \infty, \end{aligned} \tag{2.12}$$

which also shows that $\|U_\theta f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\Gamma_{a(\theta)})}$ and U_θ is one-to-one. It remains to show that U_θ is onto. For $g \in L^2(\mathbb{R}^n)$, let $G(w) = J_\theta(\phi_\theta^{-1}(w))^{-1/2} g(\phi_\theta^{-1}(w))$, $w \in \Gamma_{a(\theta)}$. We can follow (2.12) to derive

$$\int_{\Gamma_{a(\theta)}} |G(w)|^2 \, |dw| = \int_{\mathbb{R}^n} |g(x)|^2 \, dx.$$

Hence $G \in L^2(\Gamma_{a(\theta)})$, then we conclude that U_θ is onto since $U_\theta G = g$. □

3. RESONANCES

We will follow Aguilar-Balslev-Combes-Simon theory to define the resonances of $H \equiv -\Delta + V$, see [HS, §16, §18] and those resonances in a conic neighborhood of the real axis can be identified with the eigenvalues of certain non-self-adjoint operators associated with H. Using the analytic vectors $\mathcal{A}$, we recall the definition:

Definition 4. *The resonances of H associated with analytic vectors $\mathcal{A}$ are the poles of the meromorphic continuations of all matrix elements $\langle f, R_H(z)g\rangle$ ($R_H(z)$ denotes the resolvent of H), $f, g \in \mathcal{A}$, from $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ to $\{z \in \mathbb{C} : \operatorname{Im} z \leq 0\}$.*

First we introduce the spectral deformed Schrödinger operators $H(\theta)$ of H associated with the spectral deformation family $\mathcal{U} = \{U_\theta : \theta \in D_{\beta_0}\}$. Consider, for $\theta \in D_{\beta_0} \cap \mathbb{R}$,

$$H(\theta) := U_\theta H U_\theta^{-1} = p_\theta^2 + V(\phi_\theta(x)), \tag{3.1}$$

where

$$p_\theta^2 = U_\theta p^2 U_\theta^{-1}, \quad p_j \equiv \frac{1}{i}\frac{\partial}{\partial x_j}. \tag{3.2}$$

In view of Proposition 3, we can extend $H(\theta)$ to $\theta \in D_{\beta_0}$. We recall the following basic facts about p_θ^2, $\theta \in D_{\beta_0}$ in [HS, §18]:

Proposition 5. *Let p_θ^2 be as defined in (3.2), then p_θ^2, $\theta \in D_{\beta_0}$ is an analytic family of operators with domain $D(p_\theta^2) = H^2(\mathbb{R}^n)$. For the spectrum, we have $\sigma(p_\theta^2) = \sigma_{ess}(p_\theta^2) = e^{-2ia(\theta)}[0, \infty)$.*

And for the resolvent $R_\theta(z) := (p_\theta^2 - z)^{-1}$ we have:

Proposition 6. *For $\delta > 0$, we have*

$$R_\theta(z) = \mathcal{O}_\delta(1/|z|) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad -2a(\theta) + \delta < \arg z < 2\pi - 2a(\theta) - \delta. \tag{3.3}$$

Proof. We note that in the notation of Proposition 3,

$$p_\theta^2 = U_\theta(-\Delta_{a(\theta)})U_\theta^{-1}, \tag{3.4}$$

where $-\Delta_{a(\theta)} : H^2(\Gamma_{a(\theta)}) \to L^2(\Gamma_{a(\theta)})$ is defined as the restriction of Δ_z to the totally real submanifold $\Gamma_{a(\theta)}$, see [DyZ2, §4.5]. Since $U_\theta : L^2(\Gamma_{a(\theta)}) \to L^2(\mathbb{R}^n)$ and $U_\theta^{-1} : L^2(\mathbb{R}^n) \to L^2(\Gamma_{a(\theta)})$ are both isometries, we have

$$\|(p_\theta^2 - z)^{-1}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = \|(-\Delta_{a(\theta)} - z)^{-1}\|_{L^2(\Gamma_{a(\theta)})\to L^2(\Gamma_{a(\theta)})}, \quad z \notin e^{-2ia(\theta)}[0,\infty),$$

and thus (3.3) is a direct consequence of [DyZ2, Theorem 4.35]. □

Then we introduce some preliminary properties of the spectrum of $H(\theta)$:

Proposition 7. *There exists $R > 0$ such that for any $\theta \in D_{\beta_0}$, we have*

$$\sigma(H(\theta)) \cap i(R,\infty) = \emptyset.$$

As for the essential spectrum $\sigma_{ess}(H(\theta))$, we have more precisely,

$$\sigma_{ess}(H(\theta)) = e^{-2ia(\theta)}[0,\infty).$$

Remark: In fact, $\sigma(H(\theta)) \cap \{z : 0 < \arg z < 2\pi - 2a(\theta)\}$ is discrete and lies in $(-\infty, 0)$, which is a consequence of the following Lemma 1.

Proof. For $\theta \in D_{\beta_0}$, we have

$$(H(\theta) - z)R_\theta(z) = I + V(\phi_\theta(x))R_\theta(z). \tag{3.5}$$

Now assume $z \in i(R,\infty)$, note that $\theta \in D_{\beta_0} \implies -\beta_0 < a(\theta) < \beta_0$, we have

$$-2a(\theta) + \pi/4 < \arg z < 2\pi - 2a(\theta) - \pi/4, \quad \text{for all } \theta \in D_{\beta_0}.$$

Using (3.3), we see that $R_\theta(z) = \mathcal{O}(1/|z|) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ for all $z \in i(R,\infty)$ and $\theta \in D_{\beta_0}$. Recalling $\phi_\theta(\mathbb{R}^n) \subset \mathbb{R}^n \cup C^R_{\beta_0}$ and $V \in L^\infty(\mathbb{R}^n \cup C^R_{\beta_0})$, we conclude that

$$\sup_{z\in i(R,\infty)} \|V(\phi_\theta(x))R_\theta(z)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = \mathcal{O}(R^{-1}), \quad \text{uniformly for } \theta \in D_{\beta_0}.$$

Then for $R \gg 1$, $I + V(\phi_\theta(x))R_\theta(z)$ is invertible using the Neumann series:

$$(I + V(\phi_\theta(x))R_\theta(z))^{-1} = \sum_{j=0}^{\infty} (V(\phi_\theta(x))R_\theta(z))^j.$$

Hence $H(\theta) - z$ is invertible by (3.5), for all $z \in i(R,\infty)$.

For the essential spectrum $\sigma_{\mathrm{ess}}(H(\theta))$, note that $\sigma_{\mathrm{ess}}(p_\theta^2) = e^{-2ia(\theta)}[0,\infty)$ in Proposition 5, by the invariance under compact perturbations, it suffices to show that $V(\phi_\theta(x))$ is p_θ^2-compact, i.e. $V(\phi_\theta(x)) : D(p_\theta^2) = H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact. Since $H^2(B_{\mathbb{R}^n}(0,R)) \Subset L^2(B_{\mathbb{R}^n}(0,R))$, $\forall\, R > 0$, and $V \circ \phi_\theta \in L^\infty(\mathbb{R}^n)$, $V(\phi_\theta(x)) \to 0$, $x \to \infty$ by (1.1), it is easy to see the compactness of $V(\phi_\theta(x)) :\ H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. □

Now we state the main result in this section, in which we identify the resonances defined in Definition 4 as the eigenvalues of certain spectral deformed operators $H(\theta)$.

Lemma 1. *Let $H = -\Delta + V$ be a self-adjoint Schrödinger operator with a real-valued potential V satisfying our assumptions as in §1. Then for any $\theta \in D_{\beta_0} \cap \mathbb{C}^+$, we have:*

- *For $f, g \in \mathcal{A}$, the function*
$$F_{f,g}(z) \equiv \langle f, R_H(z) g \rangle, \tag{3.6}$$
defined for $\operatorname{Im} z > 0$, has a meromorphic continuation across $[0,\infty)$ into $S_\theta^- \equiv \mathbb{C} \setminus e^{-2ia(\theta)}[0,\infty)$.
- *The poles of the meromorphic continuations of all matrix elements $F_{f,g}(z)$ into S_θ^- are eigenvalues of the operator $H(\theta)$.*

Proof. With $F_{f,g}(z)$ defined in (3.6), the assumption on V implies that $F_{f,g}$ is analytic on $\mathbb{C}^+ \equiv \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Fix $z \in \mathbb{C}^+$. For $\theta \in D_{\beta_0} \cap \mathbb{R}$, U_θ is unitary and thus we can write
$$F_{f,g}(z) = \langle U_\theta f, (U_\theta R_H(z) U_\theta^{-1}) U_\theta g \rangle = \langle U_\theta f, R_{H(\theta)}(z) U_\theta g \rangle. \tag{3.7}$$
Proposition 7 implies that $\theta \in D_{\beta_0} \to R_{H(\theta)}(z)$ is an analytic map provided $z \in i(R,\infty)$. Since we can write $U_{\bar\theta} f$ instead of $U_\theta f$ in (3.7), we have
$$\theta \in D_{\beta_0} \to F_{f,g}(z;\theta) \equiv \langle U_{\bar\theta} f, R_{H(\theta)}(z) U_\theta g \rangle \tag{3.8}$$
is an analytic map provided $z \in i(R,\infty)$. Hence for any $z \in i(R,\infty)$, we have
$$F_{f,g}(z;\theta) = F_{f,g}(z), \quad \forall\, \theta \in D_{\beta_0},$$
since this is true for all $\theta \in D_{\beta_0} \cap \mathbb{R}$. Now fix any $\theta \in D_{\beta_0} \cap \mathbb{C}^+$, Proposition 7 guarantees that $F_{f,g}(z;\theta)$ can be meromorphically continued from $i(R,\infty)$ to S_θ^- since $\sigma_{\mathrm{ess}}(H(\theta)) \cap S_\theta^- = \emptyset$. We have shown that $F_{f,g}(z;\theta) = F_{f,g}(z)$, $z \in i(R,\infty)$, then by the identity principle for meromorphic functions, we conclude that $F_{f,g}(z;\theta)$ is a meromorphic continuation of $F_{f,g}(z)$ from $\mathbb{C}^+$ to S_θ^-.

Recalling that $F_{f,g}(z;\theta) = \langle U_{\bar\theta} f, (H(\theta) - z)^{-1} U_\theta g \rangle$ and that $U_\theta \mathcal{A}$, $U_{\bar\theta} \mathcal{A}$ are both dense in $L^2(\mathbb{R}^n)$, thus if $H(\theta)$ has an eigenvalue at $\lambda_\theta \in S_\theta^-$, there must exist $f, g \in \mathcal{A}$ such that λ_θ is a pole of $F_{f,g}(z;\theta)$. Conversely, if $F_{f,g}(z;\theta)$ has a pole $\lambda_\theta \in S_\theta^-$, then it must be an eigenvalue of $H(\theta)$. □

Remark: For nonzero resonance λ of H, we can define its multiplicity as the (algebraic) multiplicity of λ as an eigenvalue of some $H(\theta)$. More precisely, let $\lambda \in \{z : -2\beta_0 < \arg z < 3\pi/2 + 2\beta_0\}$ be a resonance of H, there exists $\theta \in D_{\beta_0} \cap \mathbb{C}^+$ such that $-2a(\theta) < \arg \lambda$. Lemma 1 implies that λ is also an eigenvalue of $H(\theta)$, then we define the multiplicity of resonance λ as follows:
$$m(\lambda) := m_\theta(\lambda) \equiv -\frac{1}{2\pi i} \operatorname{tr} \oint_\lambda (H(\theta) - z)^{-1} dz, \tag{3.9}$$
where the integral is over a positively oriented circle enclosing λ and containing no eigenvalues of $H(\theta)$ other than λ. To see that the multiplicity $m(\lambda)$ is well-defined,

we need to show that $m(\lambda)$ does not depend on the choice of θ. Assume $\theta_0, \theta_1 \in D_{\beta_0}$ satisfy $-2a(\theta_0) \leq -2a(\theta_1) < \arg\lambda$, let $\theta_t = (1-t)\theta_0 + t\theta_1$ then $-2a(\theta_t) < \arg\lambda$ for all $t \in [0,1]$. Let C_λ be a positively oriented circle enclosing λ with sufficiently small radius such that $C_\lambda \subset \{z : \arg z > -2a(\theta_1)\}$ and contains no resonances of H other than λ. Therefore, C_λ contains no eigenvalues of $H(\theta_t)$ other than λ for all $t \in [0,1]$ as a consequence of Lemma 1. Now we have

$$m_{\theta_t}(\lambda) = -\frac{1}{2\pi i}\operatorname{tr}\int_{C_\lambda}(H(\theta_t)-z)^{-1}dz, \quad t \in [0,1].$$

Hence $m_{\theta_t}(\lambda)$ depends continuously on t which implies that $m_{\theta_t}(\lambda)$ must be a constant as it is integer-valued. In particular, we have $m_{\theta_0}(\lambda) = m_{\theta_1}(\lambda)$, thus $m(\lambda)$ is well-defined.

4. Eigenvalues and complex scaling

In this section we will show that the eigenvalues of $H_\varepsilon \equiv -\Delta - i\varepsilon x^2 + V$ are invariant under complex scaling, in other words, these eigenvalues are the same as the eigenvalues of

$$H_\varepsilon(\theta) := U_\theta H_\varepsilon U_\theta^{-1} = p_\theta^2 - i\varepsilon\phi_\theta(x)^2 + V(\phi_\theta(x)), \quad \theta \in D_{\beta_0} \cap \mathbb{C}^+.$$

First we recall some basic properties about the Davies harmonic oscillator and its deformation, see [Z2, §3] for details. The operator $H_{\varepsilon,\gamma} := -\Delta + e^{-i\gamma}\varepsilon x^2$, $\varepsilon > 0$, $0 \leq \gamma < \pi$, was used by Davies [Da1] to illustrate properties of non-normal differential operators. We are more interested in the deformations of $H_{\varepsilon,\gamma}$ under complex scaling. Let

$$Q_{\varepsilon,\theta} = -\Delta_\theta - i\varepsilon x_\theta^2, \quad \text{where } x_\theta = z|_{\Gamma_\theta}$$

be a deformed operator on Γ_θ as in [Z2, §3]. In view of (3.4), we have

$$p_\theta^2 - i\varepsilon\phi_\theta(x)^2 = U_\theta Q_{\varepsilon,a(\theta)}U_\theta^{-1}, \quad \theta \in D_{\beta_0}. \tag{4.1}$$

Hence we can study the spectrum and the resolvents of $p_\theta^2 - i\varepsilon\phi_\theta(x)^2$ using the relevant results about $Q_{\varepsilon,a(\theta)}$. We recall [Z2, Lemma 4.] that $\sigma(Q_{\varepsilon,a(\theta)}) = \sqrt{\varepsilon}e^{-i\pi/4}(n + 2|\mathbb{N}_0^n|)$ for $\theta \in D_{\beta_0}$, then by (4.1) we have

Proposition 8. *For $\theta \in D_{\beta_0}$, $\varepsilon > 0$, the spectrum of $p_\theta^2 - i\varepsilon\phi_\theta(x)^2$ is independent of θ and given by $\sqrt{\varepsilon}e^{-i\pi/4}(n + 2|\mathbb{N}_0^n|)$.*

For the resolvents of $p_\theta^2 - i\varepsilon\phi_\theta(x)^2$:

$$R_{\varepsilon,\theta}(z) := (p_\theta^2 - i\varepsilon\phi_\theta(x)^2 - z)^{-1}, \quad \theta \in D_{\beta_0}, \tag{4.2}$$

we recall [Z2, Lemma 5.] that for $\delta > 0$, $-\pi/8 < \theta < \pi/8$, we have

$$(Q_{\varepsilon,\theta} - z)^{-1} = \mathcal{O}_\delta(1/|z|) : L^2(\Gamma_\theta) \to L^2(\Gamma_\theta), \quad -2\theta + \delta < \arg z < 3\pi/2 + 2\theta - \delta,$$

uniformly for $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0 > 0$ is a constant. Using (4.1), we have

Proposition 9. *Let $\theta \in D_{\beta_0}$, $\delta > 0$, then uniformly for $0 < \varepsilon < \varepsilon_0$, we have*

$$R_{\varepsilon,\theta}(z) = \mathcal{O}_\delta(1/|z|) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad -2a(\theta)+\delta < \arg z < 3\pi/2+2a(\theta)-\delta. \tag{4.3}$$

Now we state the main result about the eigenvalues of H_ε :

Lemma 2. *For any $\theta \in D_{\beta_0}$, $0 < \varepsilon < \varepsilon_0$,*

$$z \mapsto R_{H_\varepsilon(\theta)}(z) \equiv (H_\varepsilon(\theta) - z)^{-1}, \quad -\pi/4 < \arg z < 7\pi/4,$$

is a meromorphic family of operators on $L^2(\mathbb{R}^n)$ with poles of finite rank. Furthermore, the poles of $(H_\varepsilon(\theta) - z)^{-1}$ do not depend on $\theta \in D_{\beta_0}$ and coincide, with agreement of multiplicities, with the poles of $(H_\varepsilon - z)^{-1}$.

Proof. For fixed $\theta \in D_{\beta_0}$, one can compute

$$(H_\varepsilon(\theta) - z)R_{\varepsilon,\theta}(z) = I + V(\phi_\theta(x))R_{\varepsilon,\theta}(z),$$

then we obtain from (4.3) that

$$\begin{gathered} R_{H_\varepsilon(\theta)}(z) = R_{\varepsilon,\theta}(z)(I + V(\phi_\theta(x))R_{\varepsilon,\theta}(z))^{-1}, \\ -2a(\theta) + \delta < \arg z < 3\pi/2 + 2a(\theta) - \delta, \quad |z| \gg 1, \end{gathered} \tag{4.4}$$

where for large $|z|$, $I + V(\phi_\theta(x))R_{\varepsilon,\theta}(z)$ is invertible by a Neumann series argument. Note that $R_{\varepsilon,\theta}(z) : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$, $\arg z \neq -\pi/4$ by Proposition 8, recalling that $V(\phi_\theta(x)) : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact (see the proof of Proposition 7), we conclude that $z \mapsto V(\phi_\theta(x))R_{\varepsilon,\theta}(z)$ is an analytic family of compact operators for $-\pi/4 < z < 7\pi/4$. Hence $z \mapsto (I + V(\phi_\theta(x))R_{\varepsilon,\theta}(z))^{-1}$ is a meromorphic family of operators in the same range of z. In particular $z \mapsto R_{H_\varepsilon(\theta)}(z)$, $-\pi/4 < z < 7\pi/4$ is a meromorphic family of operators on $L^2(\mathbb{R}^n)$ with poles of finite rank.

The poles and their multiplicities are independent of θ. For that we modify the proof of Lemma 1 and define matrix elements:

$$G_{f,g}(z) = \langle f, (H_\varepsilon - z)^{-1} g\rangle,$$

and

$$G_{f,g}(z;\theta) = \langle U_{\bar\theta} f, (H_\varepsilon(\theta) - z)^{-1} U_\theta g\rangle,$$

for all $f, g \in \mathcal{A}$.

Note that $-2a(\theta)+\pi/4 < \pi/2 < 3\pi/2+2a(\theta)-\pi/4$ since $-\beta_0 < a(\theta) < \beta_0$, $\theta \in D_{\beta_0}$, using (4.3) and Neumann series argument, $H_\varepsilon(\theta) - z$ is invertible at $z = i\rho$, $\rho \gg 1$ for each $\theta \in D_{\beta_0}$. Like (3.8), we have

$$\theta \in D_{\beta_0} \to G_{f,g}(z;\theta) \equiv \langle U_{\bar\theta} f, R_{H(\theta)}(z) U_\theta g\rangle$$

is an analytic map provided $z = i\rho$, $\rho \gg 1$. Hence we have

$$G_{f,g}(z;\theta) = G_{f,g}(z), \quad \forall\, \theta \in D_{\beta_0}, \quad z = i\rho, \ \rho \gg 1, \tag{4.5}$$

since this is true for all $\theta \in D_{\beta_0} \cap \mathbb{R}$. Now fix any $\theta \in D_{\beta_0}$, note that $G_{f,g}(z)$ and $G_{f,g}(z;\theta)$ are both meromorphic in $-\pi/4 < z < 7\pi/4$, we conclude that

$$G_{f,g}(z;\theta) = G_{f,g}(z), \quad -\pi/4 < z < 7\pi/4, \tag{4.6}$$

by (4.5) and the identity principle of meromorphic functions.

Now argue as in the end of the proof of Lemma 1: if $(H_\varepsilon - z)^{-1}$ has a pole at $\lambda_\theta \in \mathbb{C} \setminus e^{-i\pi/4}[0,\infty)$, then there must exist $f, g \in \mathcal{A}$ such that λ_θ is a pole of $G_{f,g}(z;\theta)$, by (4.6), λ_θ is also a pole of $G_{f,g}(z)$ thus $(H_\varepsilon(\theta) - z)^{-1}$ must have a pole at λ_θ and vise versa. Hence for any $\theta \in D_{\beta_0}$, the poles of $(H_\varepsilon(\theta) - z)^{-1}$ in $\mathbb{C} \setminus e^{-i\pi/4}[0,\infty)$ coincide the poles of $(H_\varepsilon - z)^{-1}$ in $\mathbb{C} \setminus e^{-i\pi/4}[0,\infty)$.

To show the agreement of multiplicities, for any pole λ of $(H_\varepsilon(\theta) - z)^{-1}$, the multiplicity of λ is defined by

$$m_{\varepsilon,\theta}(\lambda) = -\frac{1}{2\pi i} \operatorname{tr} \oint_\lambda (H_\varepsilon(\theta) - z)^{-1} dz,$$

where the integral is over a positively oriented circle independent of θ enclosing λ and containing no poles other than λ. Since $m_{\varepsilon,\theta}(\lambda)$ is continuous on $\theta \in D_{\beta_0}$ and integer-valued, it must be independent of $\theta \in D_{\beta_0}$. Hence we have

$$m_{\varepsilon,\theta}(\lambda) = m_{\varepsilon,0}(\lambda) = -\frac{1}{2\pi i} \operatorname{tr} \oint_\lambda (H_\varepsilon - z)^{-1} dz$$

which is the multiplicity of λ as a pole of $(H_\varepsilon - z)^{-1}$. □

5. Meromorphic continuation

In this section we will introduce a new way to express the meromorphic continuations of resolvents $R_{H(\theta)}(z)$ and $R_{H_\varepsilon(\theta)}(z)$ in a given region $\Omega \Subset \{z : -2a(\theta) < \arg z < 3\pi/2 + 2a(\theta)\}$, which is crucial in the proof of Theorem 1. For that we will first review some properties about $R_\theta(z)$ and the weighted L^2 space, $\langle x \rangle^{-2} L^2(\mathbb{R}^n)$.

Lemma 3. *Let $\langle x \rangle^{-2} L^2(\mathbb{R}^n)$ be a weighted L^2 space with the norm*

$$\|u\|_{\langle x \rangle^{-2} L^2(\mathbb{R}^n)} = \|\langle x \rangle^2 u\|_{L^2(\mathbb{R}^n)}. \tag{5.1}$$

Then $H^2(\mathbb{R}^n) \cap \langle x \rangle^{-2} L^2(\mathbb{R}^n)$ is compactly embedded in $L^2(\mathbb{R}^n)$.

Proof. Let $u_n \in H^2(\mathbb{R}^n) \cap \langle x \rangle^{-2} L^2(\mathbb{R}^n)$ with $\|u_n\|_{H^2(\mathbb{R}^n)} \le 1$ and $\|\langle x \rangle^2 u_n\|_{L^2(\mathbb{R}^n)} \le 1$. For some $r > 0$ to be decided, we have

$$\int_{|x| \ge r} |u_n(x)|^2 dx \le \langle r \rangle^{-4} \int_{|x| \ge r} \langle x \rangle^4 |u_n(x)|^2 dx \le \langle r \rangle^{-4} \|\langle x \rangle^2 u_n\|^2_{L^2(\mathbb{R}^n)} = \langle r \rangle^{-4}.$$

Then we choose r sufficientlt large such that $\int_{|x|\geq r} |u_n(x)|^2 dx < 1/8$ for all n. Since $H^2(B(0,r) \Subset L^2(B(0,r)$, there exists subsequence $\{u_n^{(1)}\} \subset \{u_n\}$ satisfying

$$\int_{B(0,r)} |u_n^{(1)}(x) - u_m^{(1)}(x)|^2 dx < 1/2, \quad \text{for all } n, m.$$

Hence we have

$$\begin{aligned} \|u_n^{(1)} - u_m^{(1)}\|^2_{L^2(\mathbb{R}^n)} &= \int_{B(0,r)} |u_n^{(1)}(x) - u_m^{(1)}(x)|^2 dx + \int_{|x|\geq r} |u_n^{(1)}(x) - u_m^{(1)}(x)|^2 dx \\ &< 1/2 + \int_{|x|\geq r} (2|u_n^{(1)}(x)|^2 + 2|u_m^{(1)}(x)|^2) dx \\ &< 1/2 + 2/8 + 2/8 = 1. \end{aligned}$$

By the same argument, we can find $\{u_n^{(1)}\} \supset \cdots \supset \{u_n^{(j)}\} \supset \cdots$ with

$$\|u_n^{(j)} - u_m^{(j)}\|_{L^2(\mathbb{R}^n)} < 1/j, \quad \text{for all } n, m.$$

Then the subsequence $\{u_j^{(j)}\} \subset \{u_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. $\square$

Lemma 4. *Fix $\theta \in D_{\beta_0} \cap \mathbb{C}^+$, $R_\theta(z)$ is an analytic family of operators $\langle x \rangle^{-2} L^2 \to \langle x \rangle^{-2} L^2$ for $-2a(\theta) < \arg z < 2\pi - 2a(\theta)$. Furthermore, if $\Omega \Subset \{z : -2a(\theta) < \arg z < 2\pi - 2a(\theta)\}$ then there exists $C = C_{\Omega,\theta}$ such that*

$$\|R_\theta(z)\|_{\langle x \rangle^{-2} L^2 \to \langle x \rangle^{-2} L^2} \leq C, \quad z \in \Omega.$$

Proof. In view of (2.1) and (2.9), we have

$$|x|/2 < |\phi_\theta(x)| = |1 + \theta h(|x|)||x| < 3|x|/2 \implies \langle x \rangle/2 < \langle \phi_\theta(x) \rangle < 3\langle x \rangle/2.$$

Then it is equivalent to prove the lemma with $\langle \phi_\theta(x) \rangle$ replacing $\langle x \rangle$. We recall Proposition 3 to write

$$\langle \phi_\theta(x) \rangle^2 R_\theta(z) \langle \phi_\theta(x) \rangle^{-2} = U_{a(\theta)} \langle x_{a(\theta)} \rangle^2 (-\Delta_{a(\theta)} - z)^{-1} \langle x_{a(\theta)} \rangle^{-2} U_{a(\theta)}^{-1},$$

where $x_{a(\theta)}$ is the coordinate on $\Gamma_{a(\theta)}$. Then it suffices to show that, for any $0 < \alpha < \beta_0$,

$$\langle w \rangle^2 (-\Delta_\alpha - \lambda^2)^{-1} \langle w \rangle^{-2} : L^2(\Gamma_\alpha) \to L^2(\Gamma_\alpha), \ \ \mathrm{Im}(e^{i\alpha}\lambda) > 0, \tag{5.2}$$

is analytic with uniformly bounded norm provided λ in any compact subset of $\{\lambda \in \mathbb{C} : \ Im(e^{i\alpha}\lambda)\}$, where w denotes the coordinate on Γ_α. To prove (5.2), consider the integral kernel of that operator:

$$K(\lambda, w_1, w_2) = \langle w_1 \rangle^2 R_0(\lambda, w_1, w_2) \langle w_2 \rangle^{-2}, \ \ w_1, w_2 \in \Gamma_\alpha, \tag{5.3}$$

where $R_0(\lambda, w_1, w_2)$ is the integral kernel of $(-\Delta_\alpha - \lambda^2)^{-1} : L^2(\Gamma_\alpha) \to L^2(\Gamma_\alpha)$. It is easy to see that

$$\begin{aligned} |K(\lambda, w_1, w_2)| &\leq (1 + |w_1|^2)\, |R_0(\lambda, w_1, w_2)|\, \langle w_2\rangle^{-2} \\ &\leq 2(1 + |w_1 - w_2|^2 + |w_2|^2)\,(1 + |w_2|^2)^{-1}\, |R_0(\lambda, w_1, w_2)| \\ &\leq 2(1 + |w_1 - w_2|^2)|R_0(\lambda, w_1, w_2)|. \end{aligned} \tag{5.4}$$

To introduce the explicit formula of $R_0(\lambda, w_1, w_2)$, we recall that one can define $((w_1 - w_2) \cdot (w_1 - w_2))^{1/2}$ for $w_1, w_2 \in \Gamma_\alpha$, see [DyZ2, §4.5]. Then we can write

$$R_0(\lambda, w_1, w_2) = C_n \lambda^{n-2}(\lambda((w_1-w_2)\cdot(w_1-w_2))^{1/2})^{-\frac{n-2}{2}} H^{(1)}_{\frac{n}{2}-1}(\lambda((w_1-w_2)\cdot(w_1-w_2))^{1/2})$$

where $H^{(1)}_k$ denote the Hankel functions of the first kind, and we can estimate $|R_0(\lambda, w_1, w_2)|$ as follows:

$$|R_0(\lambda, w_1, w_2)| \leq \frac{P_n(\lambda((w_1 - w_2) \cdot (w_1 - w_2))^{1/2})}{(((w_1 - w_2) \cdot (w_1 - w_2))^{1/2})^{n-2}} e^{-\operatorname{Im} \lambda((w_1-w_2)\cdot(w_1-w_2))^{1/2}} \tag{5.5}$$

where P_n is a polynomial of degree $(n - 3)/2$, see [GaSm, §2.2] and [DyZ2, §4.5] for details. Using (2.11), it is easy to see that for any δ small, there exists $C_\delta > 0$ such that $|\arg((w_1 - w_2) \cdot (w_1 - w_2))^{1/2} - \alpha| < \delta$ provided $|w_1 - w_2| > C_\delta$. Note that $0 < \arg \lambda + \alpha < \pi$, for every λ, we can choose $\delta = \delta_\lambda$ such that $2\delta < \arg \lambda + \alpha < \pi - 2\delta$, then for $|z - w| > C_\lambda$, we have

$$\delta < \arg \lambda((w_1 - w_2) \cdot (w_1 - w_2))^{1/2} < \pi - \delta,$$

and thus

$$e^{-\operatorname{Im} \lambda((w_1-w_2)\cdot(w_1-w_2))^{1/2}} < e^{-c_\lambda |w_1-w_2|}, \; c_\lambda > 0, \quad \text{if } |w_1 - w_2| > C_\lambda. \tag{5.6}$$

Then using (5.4), (5.5) and (5.6), we conclude that

$$\sup_{w_1 \in \Gamma_\alpha} \int_{\Gamma_\alpha} |K(\lambda, w_1, w_2)| dw_1 < M_\lambda, \quad \sup_{w_2 \in \Gamma_\alpha} \int_{\Gamma_\alpha} |K(\lambda, w_1, w_2)| dz < M_\lambda.$$

By Schur criterion, we proved (5.2), the analyticity in λ is easy to see using the explicit formula of $R_0(\lambda, w_1, w_2)$. If $\lambda \in K \Subset \{\lambda \in \mathbb{C} : \; Im(e^{i\alpha}\lambda)\}$, then there exist c_K and C_K such that

$$e^{-\operatorname{Im} \lambda((w_1-w_2)\cdot(w_1-w_2))^{1/2}} < e^{-c_K |w_1-w_2|}, \; c_K > 0, \quad \text{if } |w_1 - w_2| > C_K.$$

Follow the above argument, there exists $M = M_K > 0$ such that

$$\|\langle w\rangle^2(-\Delta_\alpha - \lambda^2)^{-1}\langle w\rangle^{-2}\|_{L^2(\Gamma_\alpha)\to L^2(\Gamma_\alpha)} < M_K, \quad \text{for all } \lambda \in K,$$

which completes the proof. □

Now we state the main result of this section:

Lemma 5. *Fix any* $\theta \in D_{\beta_0} \cap \mathbb{C}^+$ *and* $\Omega \Subset \{z : -2a(\theta) < \arg z < 3\pi/2 + 2a(\theta)\}$ *, there exists* $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$*,* $\chi \equiv 1$ *on* $B(0,T)$ *for some* $T > 0$ *such that for* $0 \le \varepsilon < \varepsilon_0$*,* $H_\varepsilon(\theta) - \chi V - z$ *is invertible in* Ω *and*

$$z \mapsto (I + R_{H_\varepsilon(\theta)-\chi V}(z)\,\chi V)^{-1}, \quad z \in \Omega,$$

is a meromorphic family of operators on $L^2(\mathbb{R}^n)$ *with poles of finite rank, where we write* $R_{H_\varepsilon(\theta)-\chi V}(z) = (H_\varepsilon(\theta) - \chi V - z)^{-1}$ *for simplicity. Moreover,*

$$m_{\varepsilon,\theta}(z) := \frac{1}{2\pi i} \operatorname{tr} \oint_z (I + R_{H_\varepsilon(\theta)-\chi V}(w)\,\chi V)^{-1} \partial_w (R_{H_\varepsilon(\theta)-\chi V}(w)\,\chi V) dw, \tag{5.7}$$

where the integral is over a positively oriented circle enclosing z *and containing no poles other than possibly* z*, satisfies*

$$m_{\varepsilon,\theta}(z) = \frac{1}{2\pi i} \operatorname{tr} \oint_z (w - H_\varepsilon(\theta))^{-1} dw, \quad 0 \le \varepsilon < \varepsilon_0, \tag{5.8}$$

where $H_0(\theta) = H(\theta)$*.*

Proof. We modify the argument in [Z2, §4] to our setting. First there exists $\delta = \delta_\Omega$ such that $\Omega \subset \mathcal{C}_\delta := \{z : -2a(\theta) + \delta < \arg z < 3\pi/2 + 2a(\theta) - \delta,\ |z| > \delta\}$, we recall (3.3) and (4.3) that uniformly for $0 \le \varepsilon < \varepsilon_0$, we have

$$R_{\varepsilon,\theta}(z) = \mathcal{O}_\delta(1/|z|) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad z \in \mathcal{C}_\delta. \tag{5.9}$$

Hence $\|R_{\varepsilon,\theta}(z)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} < C_\delta$, $\forall z \in \mathcal{C}_\delta$, for some $C_\delta > 0$. In view of (1.1), for T sufficiently large, we have $\|(1-\chi)V\|_{L^\infty} < 1/2C_\delta$ and thus

$$\|R_{\varepsilon,\theta}(z)(1-\chi)V\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} < 1/2, \quad \text{for all } z \in \mathcal{C}_\delta. \tag{5.10}$$

Then $(I + R_{\varepsilon,\theta}(z)(1-\chi)V)$ is invertible by the Neumann series argument, which implies that $H_\varepsilon(\theta) - \chi V - z$ is invertible and

$$R_{H_\varepsilon(\theta)-\chi V}(z) = (H_\varepsilon(\theta) - \chi V - z)^{-1} = (I + R_{\varepsilon,\theta}(z)(1-\chi)V)^{-1} R_{\varepsilon,\theta}(z), \quad \forall z \in \mathcal{C}_\delta. \tag{5.11}$$

Since $\chi V \in L^\infty(\mathbb{R}^n)$, (5.9) and (5.11) imply that for $z \in \mathcal{C}_\delta$, $|z| \gg 1$, both $I + \chi V R_{H_\varepsilon(\theta)-\chi V}(z)$ and $I + R_{H_\varepsilon(\theta)-\chi V}(z)\chi V$ are invertible by the Neumann series argument. Hence we have

$$\begin{aligned} R_{H_\varepsilon(\theta)}(z) &= R_{H_\varepsilon(\theta)-\chi V}(z)(I + \chi V R_{H_\varepsilon(\theta)-\chi V}(z))^{-1} \\ &= R_{H_\varepsilon(\theta)-\chi V}(z) \sum_{j=0}^\infty (-1)^j (\chi V R_{H_\varepsilon(\theta)-\chi V}(z))^j \\ &= R_{H_\varepsilon(\theta)-\chi V}(z) \left(I - \chi V \sum_{j=0}^\infty (-1)^j (R_{H_\varepsilon(\theta)-\chi V}(z)\chi V)^j R_{H_\varepsilon(\theta)-\chi V}(z) \right) \\ &= R_{H_\varepsilon(\theta)-\chi V}(z)[I - \chi V (I + R_{H_\varepsilon(\theta)-\chi V}(z)\chi V)^{-1} R_{H_\varepsilon(\theta)-\chi V}(z)], \end{aligned} \tag{5.12}$$

Using (5.11), we have

$$R_{H_\varepsilon(\theta)-\chi V}(z)\chi V = (I + R_{\varepsilon,\theta}(z)(1-\chi)V)^{-1}R_{\varepsilon,\theta}(z)\chi V.$$

For $\varepsilon > 0$, $R_{\varepsilon,\theta}(z) : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n) \cap \langle\phi_\theta(x)\rangle^{-2}L^2(\mathbb{R}^n)$, then Lemma 3 implies that $R_{\varepsilon,\theta}(z) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact. For $\varepsilon = 0$, note that $\chi V : L^2(\mathbb{R}^n) \to \langle x\rangle^{-2}L^2(\mathbb{R}^n)$, by Lemma 4 we have $R_\theta(z)\chi V : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n) \cap \langle x\rangle^{-2}L^2(\mathbb{R}^n)$, then Lemma 3 implies that $R_\theta(z)\chi V : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact. Hence we can conclude that $z \mapsto R_{H_\varepsilon(\theta)-\chi V}(z)\chi V$ is an analytic family of compact operators for $z \in \mathcal{C}_\delta,\ \ 0 \le \varepsilon < \varepsilon_0$, and thus $z \mapsto (I + R_{H_\varepsilon(\theta)-\chi V}(z)\chi V)^{-1}$ is a meromorphic family of operators in the same range of z.

Then we recall Lemma 1 and 2 that $R_{H_\varepsilon(\theta)}(z)$ is meromorphic in $-2a(\theta) < \arg z < 3\pi/2 + 2a(\theta)$, by the identity principle of meromorphic operators, we conclude that (5.12) holds for all $z \in \mathcal{C}_\delta$ in the sense of meromorphic family of operators.

To obtain the multiplicity formula, we assume that $z \in \Omega$, then there exists a neighborhood $z \in U \subset \Omega$ and finite rank operators $A_j,\ 1 \le j \le J$ such that

$$(I + R_{H_\varepsilon(\theta)-\chi V}(w)\chi V)^{-1} - \sum_{j=1}^{J}\frac{A_j}{(w-z)^j} \quad \text{is holomorphic in } w \in U.$$

Let $C_z \subset U$ be a positively oriented circle enclosing z and containing no poles of $(I + R_{H_\varepsilon(\theta)-\chi V}(w)\chi V)^{-1}$ other than possibly z, thus it also contains no poles of $(w - H_\varepsilon(\theta))^{-1}$ other than possibly z as a consequence of (5.12). On the one hand, we can compute

$$\begin{aligned}
m_{\varepsilon,\theta}(z) &= \frac{1}{2\pi i}\operatorname{tr}\int_{C_z}(I + R_{H_\varepsilon(\theta)-\chi V}(w)\chi V)^{-1}\partial_w(R_{H_\varepsilon(\theta)-\chi V}(w)\chi V)dw \\
&= \frac{1}{2\pi i}\operatorname{tr}\int_{C_z}(I + R_{H_\varepsilon(\theta)-\chi V}(w)\chi V)^{-1}R_{H_\varepsilon(\theta)-\chi V}(w)^2\chi V dw \\
&= \frac{1}{2\pi i}\operatorname{tr}\int_{C_z}\sum_{j=1}^{J}\frac{A_j R_{H_\varepsilon(\theta)-\chi V}(w)^2\chi V}{(w-z)^j}dw \\
&= \sum_{j=1}^{J}\frac{1}{(j-1)!}\operatorname{tr}\partial_z^{j-1}(A_j R_{H_\varepsilon(\theta)-\chi V}(z)^2\chi V) \\
&= \sum_{j=1}^{J}\sum_{k=0}^{j-1}\frac{1}{k!(j-1-k)!}\operatorname{tr} A_j\,\partial_z^k R_{H_\varepsilon(\theta)-\chi V}(z)\,\partial_z^{j-1-k}R_{H_\varepsilon(\theta)-\chi V}(z)\chi V.
\end{aligned} \tag{5.13}$$

On the other hand, by (5.12) we have

$$\begin{aligned}
&\frac{1}{2\pi i}\operatorname{tr}\oint_z (w-H_\varepsilon(\theta))^{-1}dw\\
&=\frac{1}{2\pi i}\operatorname{tr}\int_{C_z} R_{H_\varepsilon(\theta)-\chi V}(w)\chi V(I+R_{H_\varepsilon(\theta)-\chi V}(w)\chi V)^{-1}R_{H_\varepsilon(\theta)-\chi V}(w)dw\\
&=\frac{1}{2\pi i}\operatorname{tr}\int_{C_z}\sum_{j=1}^{J}\frac{R_{H_\varepsilon(\theta)-\chi V}(w)\chi V A_j R_{H_\varepsilon(\theta)-\chi V}(w)}{(w-z)^j}dw\\
&=\sum_{j=1}^{J}\frac{1}{(j-1)!}\operatorname{tr}\partial_z^{j-1}(R_{H_\varepsilon(\theta)-\chi V}(z)\chi V A_j R_{H_\varepsilon(\theta)-\chi V}(z))\\
&=\sum_{j=1}^{J}\sum_{k=0}^{j-1}\frac{1}{k!(j-1-k)!}\operatorname{tr}\partial_z^{j-1-k}R_{H_\varepsilon(\theta)-\chi V}(z)\chi V A_j\,\partial_z^k R_{H_\varepsilon(\theta)-\chi V}(z).
\end{aligned}\tag{5.14}$$

Now we compare (5.13) and (5.14). Since A_j factors have finite rank, we can apply cyclicity of the trace to obtain the multiplicity formula (5.8).

□

6. Proof of convergence

The proof of convergence is based on Lemma 1, Lemma 2, Lemma 5 and the following lemma:

Lemma 6. *Fix any $\theta \in D_{\beta_0}\cap\mathbb{C}^+$ and $\Omega \Subset \{z : -2a(\theta) < \arg z < 3\pi/2+2a(\theta)\}$, there exists $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\chi \equiv 1$ on $B(0,T)$ for some $T > 0$ such that for $0 < \varepsilon < \varepsilon_0$,*

$$T_{\varepsilon,\theta}(z) := (H_\varepsilon(\theta)-\chi V - z)^{-1}\phi_\theta(x)^2(H(\theta)-\chi V-z)^{-1}\chi V$$

is an analytic family of operators: $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Furthermore, there exists $C = C_{\Omega,\theta}$ such that

$$\|T_{\varepsilon,\theta}(z)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C, \quad z\in\Omega, \quad \textit{uniformly for } 0<\varepsilon<\varepsilon_0. \tag{6.1}$$

Proof. We recall the proof of Lemma 5 that for T sufficiently large, $H_\varepsilon(\theta)-\chi V - z$ is invertible, then (5.10) and (5.11) imply that

$$\|(H_\varepsilon(\theta)-\chi V-z)^{-1}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C_\Omega, \quad z\in\Omega,$$

for some $C_\Omega > 0$. Hence it suffices to prove

$$\|\phi_\theta(x)^2(H(\theta)-\chi V-z)^{-1}\chi V\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C_\Omega, \quad z\in\Omega. \tag{6.2}$$

By Lemma 4, we have $\|R_\theta(z)\|_{\langle x\rangle^{-2}L^2\to\langle x\rangle^{-2}L^2} \le C$. We can choose T sufficiently large such that (5.10) still holds and $\|(1-\chi)V\|_{L^\infty} < 1/2C$, then we have

$$\|R_\theta(z)(1-\chi V)\|_{\langle x\rangle^{-2}L^2\to\langle x\rangle^{-2}L^2} < 1/2, \quad z\in\Omega.$$

Hence $(I+R_\theta(z)(1-\chi)V)^{-1}: L^2 \to L^2$ defined by the Neumann series in the proof of Lemma 5 also maps $\langle x\rangle^{-2}L^2$ to $\langle x\rangle^{-2}L^2$ by the same Neumann series and we have

$$\|(I+R_\theta(z)(1-\chi)V)^{-1}\|_{\langle x\rangle^{-2}L^2\to\langle x\rangle^{-2}L^2} < 2, \quad z\in\Omega. \tag{6.3}$$

Since $\chi V: L^2 \to \langle x\rangle^{-2}L^2$ with the operator norm bounded by $\|\langle x\rangle^2\chi V\|_{L^\infty} = C_\Omega$, by Lemma 4, (5.11) and (6.3) we conclude that

$$\begin{aligned}
&\|(H(\theta)-\chi V-z)^{-1}\chi V\|_{L^2\to\langle x\rangle^{-2}L^2}\\
=&\|(I+R_\theta(z)(1-\chi)V)^{-1}R_\theta(z)\chi V\|_{L^2\to\langle x\rangle^{-2}L^2}\\
\le&\|(I+R_\theta(z)(1-\chi)V)^{-1}R_\theta(z)\|_{\langle x\rangle^{-2}L^2\to\langle x\rangle^{-2}L^2}\|\chi V\|_{L^2\to\langle x\rangle^{-2}L^2}\\
\le& C_\Omega,
\end{aligned}$$

which implies (6.2). ☐

Now we state the result about the convergence of eigenvalues of the deformed operator $H_\varepsilon(\theta)$:

Theorem 2. *Fix $\theta\in D_{\beta_0}\cap\mathbb{C}^+$ and $\Omega\Subset\{z: -2a(\theta)<\arg z<3\pi/2+2a(\theta)\}$, there exists $\delta_0=\delta_0(\Omega)$ satisfying the following:*

For any $0<\delta<\delta_0$ there exists $\varepsilon'>0$ such that for any $z\in\Omega$ with $m_\theta(z)>0$ and $0<\varepsilon<\varepsilon'$, $H_\varepsilon(\theta)$ has $m_\theta(z)$ eigenvalues in $B(z,\delta)$, where $m_\theta(z)$ is the multiplicity of the eigenvalue of $H(\theta)$ at z - see (3.9).

Proof. Since the eigenvalues of $H(\theta)$ are isolated and $\overline{\Omega}$ is compact, there are finite many $z\in\Omega$ with $m_\theta(z)>0$, we denote them by $z_1,\dots,z_J$. Then we can choose δ_0 such that $B(z_j,\delta_0)$, $j=1,\dots,J$ are disjoint.

Now we fix $\delta<\delta_0$, by Lemma 5, $I+R_{H(\theta)-\chi V}(w)\chi V$ is invertible in $\Omega\setminus\{z_1,\dots,z_J\}$, thus we have

$$\|(I+R_{H(\theta)-\chi V}(w)\chi V)^{-1}\|_{L^2\to L^2} < C(\delta), \quad w\in\partial B(z,\delta), \text{ for all } z\in\{z_1,\dots,z_J\},$$

for some $C(\delta)>0$. We note that in the notation of Lemma 6,

$$I+R_{H_\varepsilon(\theta)-\chi V}(w)\chi V-(I+R_{H(\theta)-\chi V}(w)\chi V) = i\varepsilon T_{\varepsilon,\theta}(w).$$

Hence there exists $0<\varepsilon'<\varepsilon_0$ such that for any $\varepsilon<\varepsilon'$,

$$\|(I+R_{H(\theta)-\chi V}(w)\chi V)^{-1}(I+R_{H_\varepsilon(\theta)-\chi V}(w)\chi V-(I+R_{H(\theta)-\chi V}(w)\chi V))\|<1$$

on $\partial B(z,\delta)$. Now we apply Gohberg-Sigal-Rouché theorem, see [GS] and [DyZ2, Appendix C.] to obtain that

$$\begin{aligned}
&\frac{1}{2\pi i}\operatorname{tr}\int_{\partial B(z,\delta)}(I+R_{H_\varepsilon(\theta)-\chi V}(w)\chi V)^{-1}\partial_w(R_{H_\varepsilon(\theta)-\chi V}(w)\chi V)dw\\
=&\frac{1}{2\pi i}\operatorname{tr}\int_{\partial B(z,\delta)}(I+R_{H(\theta)-\chi V}(w)\chi V)^{-1}\partial_w(R_{H(\theta)-\chi V}(w)\chi V)dw.
\end{aligned}$$

Then we recall (5.8) to conclude that

$$\frac{1}{2\pi i}\operatorname{tr}\int_{\partial B(z,\delta)}(w-H_\varepsilon(\theta))^{-1}dw = m_\theta(z),$$

which implies that $H_\varepsilon(\theta)$ has $m_\theta(z)$ eigenvalues in $B(0,\delta)$. □

Finally, we can give the proof of Theorem 1.

Proof. We assume from now on that $\varepsilon < \varepsilon_0$. Fix any $\Omega \Subset \{z : -2\beta_0 < \arg z < 3\pi/2 + 2\beta_0\}$, we can choose $\theta \in D_{\beta_0} \cap \mathbb{C}^+$ such that

$$\Omega \Subset \{z : -2a(\theta) < \arg z < 3\pi/2 + 2a(\theta)\}.$$

In view of Lemma 1, we see that $\{z_j\}_{j=1}^\infty$, the resonances of H in Ω, can be identified as the eigenvalues of $H(\theta)$, denoted by $\{z_{\theta,j}\}_{j=1}^\infty$. Similarly, Lemma 2 guarantees that $\{z_j(\varepsilon)\}_{j=1}^\infty$, the eigenvalues of H_ε in $\{z : -2a(\theta) < \arg z < 3\pi/2 + 2a(\theta)\}$, are the eigenvalues of $H_\varepsilon(\theta)$, denoted by $\{z_{\theta,j}(\varepsilon)\}_{j=1}^\infty$. Hence it suffices to show

$$z_{\theta,j}(\varepsilon) \to z_{\theta,j}, \quad \varepsilon \to 0+, \quad \text{uniformly on } \Omega, \tag{6.4}$$

which is a direct result of Theorem 2. □

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