Qualifying Exam Syllabus

Haoren Xiong

Major topic: Microlocal Analysis (Classical Analysis)

Fundamentals. Symbols, their basic properties, asymptotic sums. Phase functions. Oscillatory integrals, basic regularity properties, Schwartz kernel. Method of stationary phase. [GS94]

Pseudodifferential operators. Properly supported operators, the complete symbol. Adjoints, products of operators and changes of variables. Construction of a parametrix for an elliptic pseudodifferential operator. L^2 and H^s-regularity. Cotlar-Stein lemma and Calderón-Vaillancourt theorem. Sharp Gårding inequality.

Local symplectic geometry. Canonical symplectic structure on the cotangent bundle. Lagrangian submanifolds and local existence of solutions for Hamilton-Jacobi equations. Darboux theorems. Canonical transformations.

Wavefront set. Definition, basic properties and behavior under pseudodifferential operators. Operations: inner product, tensor product, distributions kernels, pullback, multiplication. Wavefront sets of oscillatory integrals.

WKB approximation for solutions of PDEs. Construction of a local parametrix for the Cauchy problem for a strictly hyperbolic PDE. Singular support of its kernel. Propagation of singularities for operators of real principal type.

Spectral theory for elliptic operators. Spectral asymptotics for elliptic operators. Powers of elliptic operators.

Major topic: Harmonic Analysis (Classical Analysis)

Fourier Series and Fourier Transform. Dirichlet and Fejer kernels. Poisson summation formula. Riemann-Lebesgue lemma. Convergence of Fourier series. Tempered distributions. Fourier transform for LCA group. [Chr18]

Hardy-Littlewood Maximal Function. Vitali covering lemma. L^p boundedness. Lebesgue differentiation theorem. Marcinkiewicz Interpolation Theorem. Weak $(1, 1)$ estimate.

BMO and Carleson Measures. Sharp function. Whitney decomposition lemma. BMO and John-Nirenberg inequality. Carleson measures. H^p and boundary values of harmonic function.

Singular Integrals of convolution type. Hilbert and Riesz transforms and their L^p boundedness. Calderón-Zygmund decomposition. General and Maximal singular integrals and L^p bounds.

Almost Orthogonality and Littlewood-Paley theory. Almost orthogonality in Hilbert spaces. Littlewood-Paley decomposition. Converse inequalities and Interpretation. Estimates for maximal operators and singular operators.

Minor topic: Riemannian Geometry (Geometry)

Differential manifolds. Basic structures: morphisms, orientation, local coordinates, partition of unity. Vector bundles and vector fields. Tensor bundles and tensor fields. Tangent and cotangent bundles, differential forms.

Riemannian metrics. Existence. Basic structures: raising and lowering indices, inner product on tensors, integration. Model examples: Euclidean spaces, spheres, hyperbolic models.

Connections and geodesics. Affine connections, Levi-Civita connection. Existence and uniqueness of geodesics, the exponential map, convex neighborhoods. Length of curves, first variation formula, minimizing properties of geodesics. Complete manifolds, Hopf-Rinow Theorem.

Curvature. Riemannian curvature tensor. Sectional curvature. Ricci and scalar curvature.

Major theorems. Gauss-Bonnet Theorem. Classification of constant curvature metrics. The theorem of Bonnet and Myers.

References

[Car92] Manfredo Perdigão do Carmo. *Riemannian geometry*. Birkhäuser, 1992.

[Chr18] Michael Christ. Euclidean harmonic analysis: Lecture notes for mathematics 258. University of California, Berkeley, 2018.

[GS94] Alain Grigis and Johannes Sjöstrand. *Microlocal analysis for differential operators: an introduction*, volume 196. Cambridge University Press, 1994.

[Ste16] Elias M Stein. *Singular integrals and differentiability properties of functions (PMS-30)*, volume 30. Princeton university press, 2016.