GENERIC SIMPLICITY OF RESONANCES IN OBSTACLE SCATTERING

HAOREN XIONG

Abstract. We show that all resonances in Dirichlet obstacle scattering (in $\mathbb{C}$ in odd dimensions and in the logarithmic cover of $\mathbb{C}\setminus\{0\}$ in even dimensions) are generically simple in the class of obstacles with C^k (and C^∞) boundaries, $k \geq 2$.

1. Introduction

The evolution of eigenvalues of second order elliptic operators under boundary perturbations have been studied through different perspectives since Hadamard [Ha08]. Uhlenbeck [Uh76] proved generic properties of eigenvalues and eigenfunctions of second order elliptic operators with respect to variation of the domain for general boundary conditions. Henry [He05] developed a general theory on perturbation of domains for second order elliptic operators. In this paper we prove that a generic boundary perturbation in obstacle scattering for the Dirichlet Laplacian splits the multiplicities of all resonances in both odd and even dimensions. We formulate the problem as follows:

Suppose that $\mathcal{O} \subset \mathbb{R}^n$ is a bounded open set such that $\partial\mathcal{O}$ is a C^2 hypersurface in $\mathbb{R}^n$. Let $\Delta_{\mathcal{O}}$ be the self-adjoint Dirichlet Laplacian on $\mathbb{R}^n \setminus \mathcal{O}$ with domain

$$\mathcal{D}(\Delta_{\mathcal{O}}) := H^2(\mathbb{R}^n \setminus \mathcal{O}) \cap H_0^1(\mathbb{R}^n \setminus \mathcal{O}). \tag{1.1}$$

The resolvent of $-\Delta_{\mathcal{O}}$,

$$R_{\mathcal{O}}(\lambda) := (-\Delta_{\mathcal{O}} - \lambda^2)^{-1} : L^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\mathbb{R}^n \setminus \mathcal{O}), \quad \operatorname{Im}\lambda > 0,$$

continues meromorphically as an operator from $e^{-|x|^2}L^2(\mathbb{R}^n \setminus \mathcal{O})$ to $e^{|x|^2}L^2(\mathbb{R}^n \setminus \mathcal{O})$, see Dyatlov–Zworski [DyZw19, §4.2] and a review in §2. When n is odd the continuation is to $\lambda \in \mathbb{C}$ and when n is even to the logarithmic cover of $\mathbb{C} \setminus \{0\}$:

$$\Lambda = \exp^{-1}(\mathbb{C} \setminus \{0\}). \tag{1.2}$$

We denote the set of poles of $R_{\mathcal{O}}(\lambda)$ by $\operatorname{Res}(\mathcal{O})$. The elements of $\operatorname{Res}(\mathcal{O})$ are called scattering resonances for the obstacle $\mathcal{O}$. We recall the following facts from [DyZw19, Theorem 4.19] (for n odd) and Christiansen [Ch17, §6] (for n even) that:

$$0 \notin \operatorname{Res}(\mathcal{O}),\ \text{for } n \text{ odd}; \quad 0 \text{ is not a limit point of } \operatorname{Res}(\mathcal{O}) \text{ for } n \text{ even}. \tag{1.3}$$

Thus for $\lambda \in \mathrm{Res}(\mathcal{O})$, its multiplicity $m_{\mathcal{O}}(\lambda)$ satisfies

$$m_{\mathcal{O}}(\lambda) := \mathrm{rank} \oint_{\lambda} R_{\mathcal{O}}(\zeta) d\zeta = \mathrm{rank} \oint_{\lambda} R_{\mathcal{O}}(\zeta) 2\zeta d\zeta, \tag{1.4}$$

where the integral is over a circle containing no other pole of $R_{\mathcal{O}}(\zeta)$ than λ, see [DyZw19, §4.2]. A resonance $\lambda \in \mathrm{Res}(\mathcal{O})$ is called simple if $m_{\mathcal{O}}(\lambda) = 1$.

To describe the deformations of obstacles, we follow Pereira [Pe04] and introduce a set of C^k–smooth mappings ($k \geq 2$) which deforms the obstacle $\mathcal{O}$:

$$\mathrm{Diff}(\mathcal{O}) := \left\{ \begin{matrix} \Phi \in C^k(\mathbb{R}^n; \mathbb{R}^n) \text{ is a } C^k\text{-diffeomorphism} : \ \Phi(\partial \mathcal{O}) = \partial\, \Phi(\mathcal{O}), \\ \text{and } \Phi(x) = x, \ \forall\, |x| > R, \quad \text{for some } R > 0. \end{matrix} \right\}. \tag{1.5}$$

Let X be the class of obstacles diffeomorphic to a fixed obstacle $\mathcal{O}_0$ (for example, $\mathcal{O}_0 = B_{\mathbb{R}^n}(0,1)$), that is,

$$X = \{\Phi(\mathcal{O}_0) : \Phi \in \mathrm{Diff}(\mathcal{O}_0)\}. \tag{1.6}$$

We introduce a topology in this set by defining a sub-basis of the neighborhoods of a given $\mathcal{O} \in X$ by

$$\{\Phi(\mathcal{O}) : \Phi \in \mathrm{Diff}(\mathcal{O}),\ \|\Phi - \mathrm{id}\,\|_{C^2(\mathbb{R}^n \setminus \mathcal{O})} < \varepsilon, \text{ with } \varepsilon \text{ sufficiently small.}\}$$

Now we state the main result of this paper, by a generic set we mean an intersection of open dense sets:

Theorem. *For any fixed obstacle $\mathcal{O}_0$ and the corresponding family X given in* (1.6)*, there exists a generic set $\mathcal{X} \subset X$ such that for every $\mathcal{O} \in \mathcal{X}$, all resonances $\lambda \in \mathrm{Res}(\mathcal{O})$ are simple.*

Remark 1: We should point out that an analogue of this result for Robin boundary condition (and in particular for the Neumann boundary condition) remains an open problem. The difficulty was overcome by Uhlenbeck [Uh72] in the case of Neumann eigenvalue problem in a bounded domain Ω by using Transversality Theorem in infinite dimensions and then deriving a contradiction from the equation $\nabla_{\partial\Omega} u \cdot \nabla_{\partial\Omega} v = \lambda uv$ on $\partial\Omega$ where $\lambda > 0$, $u, v \in C^2(\partial\Omega; \mathbb{R})$ and $uv \neq 0$ on an open dense subset of $\partial\Omega$, see also [He05, Example 6.4] for more details. In the case of obstacle scattering with Neumann boundary condition, this argument does not seem to apply for $\nabla_{\partial\Omega} u \cdot \nabla_{\partial\Omega} v = zuv$ when u, v are complex-valued and z is a complex resonance.

Remark 2: Klopp and Zworski [KlZw95] proved that a generic potential perturbation in black box scattering (for a definition see for instance [DyZw19, §4] and §2) splits the multiplicities of all resonances. This result was extended to scattering on asymptotically hyperbolic manifolds by Borthwick and Perry [BoPe02], in which the method of complex scaling used in [KlZw95] was replaced by Agmon's perturbation theory of resonances [Ag98]. We will combine the strategies of [KlZw95] and [BoPe02] in the proof of our theorem. However, the boundary perturbation produces additional difficulties.

The paper is organized as follows. In §2 we review the meromorphy of the resolvent $R_{\mathcal{O}}(\lambda)$. More precisely, we show that $R_{\mathcal{O}}(\lambda)$ admits a meromorphic continuation to $\lambda \in \mathbb{C}$ in odd dimensions, to $\lambda \in \Lambda$ in even dimensions, as an operator between weighted Hilbert spaces $e^{-|x|^2}L^2(\mathbb{R}^n \setminus \mathcal{O})$ and $e^{|x|^2}L^2(\mathbb{R}^n \setminus \mathcal{O})$ (instead of $L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$) as an preparation for applying Agmon's perturbation theory of resonances. In §3 we conjugate the Dirichelet Laplacian of the deformed obstacle $\Phi(\mathcal{O})$ by the pullback Φ^* to obtain an operator on the original domain $\mathbb{R}^n \setminus \mathcal{O}$. As a result, the variation of the domain is transferred to the coefficients of the differential operator. In §4 we review Agmon's perturbation theory of resonances [Ag98] in which the resonances are realized as eigenvalues of a non-self-adjoint operator on an abstractly constructed Banach space. We remark that the method of complex scaling is also capable of characterizing resonances in $\mathbb{C}$ in odd dimensions and resonances in Λ in even dimensions with small argument, but Agmon's method allows us to prove the generic simplicity of all resonances in the whole Λ in even dimensions. The proof of the theorem is completed in §5 by adapting the strategy in [KlZw95] to the case of boundary perturbations.

Acknowledgments. The author would like to thank Maciej Zworski for helpful discussions. This project was supported in part by the National Science Foundation grant DMS-1901462.

2. Meromorphic continuation

In this section we will follow [DyZw19, §4] to introduce a general class of compactly supported self-adjoint perturbations of the Laplacian in $\mathbb{R}^n$, P, which are called black box Hamiltonians, and show that the resolvent of P admits a meromorphic continuation to $\mathbb{C}$ when n is odd; Λ when n is even, as an operator between some weighted Hilbert spaces.

Let $\mathcal{H}$ be a complex separable Hilbert space with an orthogonal decomposition:

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)), \tag{2.1}$$

where $B(x, R) = \{y \in \mathbb{R}^n : |x-y| < R\}$ and R_0 is fixed. The corresponding orthogonal projections will be denoted by $u \mapsto u|_{B(0,R_0)}$, and $u \mapsto u|_{\mathbb{R}^n \setminus B(0,R_0)}$ or simply by the characteristic function 1_L of the corresponding set L.

We define a weighted subspace of $\mathcal{H}$ as

$$\mathcal{H}_0 := \{u \in \mathcal{H} : u|_{\mathbb{R}^n \setminus B(0,R_0)} \in e^{-|x|^2}L^2(\mathbb{R}^n \setminus B(0, R_0))\}, \tag{2.2}$$

and a larger space containing $\mathcal{H}$:

$$\mathcal{H}_1 := \mathcal{H}_{R_0} \oplus e^{|x|^2}L^2(\mathbb{R}^n \setminus B(0, R_0)). \tag{2.3}$$

We now consider an unbounded self-adjoint operator

$$P : \mathcal{H} \to \mathcal{H} \quad \text{with domain } \mathcal{D}(P). \tag{2.4}$$

Assume that

$$\mathcal{D}(P)|_{\mathbb{R}^n \setminus B(0,R_0)} \subset H^2(\mathbb{R}^n \setminus B(0,R_0)), \tag{2.5}$$

and conversely, $u \in \mathcal{D}(P)$ if $u \in H^2(\mathbb{R}^n \setminus B(0,R_0))$ and u vanishes near $B(0,R_0)$;

$$1_{B(0,R_0)}(P+i)^{-1} \text{ is compact.} \tag{2.6}$$

We also assume that,

$$1_{\mathbb{R}^n \setminus B(0,R_0)} P u = -\Delta(u|_{\mathbb{R}^n \setminus B(0,R_0)}), \qquad \text{for all } u \in \mathcal{D}(P). \tag{2.7}$$

The space $\mathcal{D}_1(P)$ is defined using (2.3),

$$\begin{aligned} \mathcal{D}_1(P) := \{u \in \mathcal{H}_1 : \chi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n),\ \chi|_{B(0,R_0)} \equiv 1 \\ \Rightarrow \chi u \in \mathcal{D}(P),\ \Delta((1-\chi)u) \in \mathcal{H}_1\}. \end{aligned} \tag{2.8}$$

We are now ready to state the main result of this section:

Proposition 2.1. *Suppose that P is a black box Hamiltonian. Then*

$$R(\lambda) := (P - \lambda^2)^{-1} : \mathcal{H} \to \mathcal{D}(P) \quad \textit{is meromorphic for } \operatorname{Im} \lambda > 0. \tag{2.9}$$

Moreover, when n is odd, the resolvent extends to a meromorphic family

$$R(\lambda) : \mathcal{H}_0 \to \mathcal{D}_1(P), \quad \lambda \in \mathbb{C}. \tag{2.10}$$

When n is even (2.10) holds with $\mathbb{C}$ replaces by the logarithmic plane Λ in (1.2).

The proof is the same as the one of [DyZw19, Theorem 4.4]. The only difference is that unlike the free resolvent $R_0(\lambda) := (-\Delta - \lambda^2)^{-1}$ mermorphically continued as an operator between $L^2_{\mathrm{comp}}(\mathbb{R}^n)$ and $H^2_{\mathrm{loc}}(\mathbb{R}^n)$ there, we have to show that

$$\begin{aligned} \lambda \mapsto R_0(\lambda) : e^{-|x|^2} L^2(\mathbb{R}^n) \to e^{|x|^2} L^2(\mathbb{R}^n), \\ \lambda \mapsto [\Delta, \chi] R_0(\lambda) : e^{-|x|^2} L^2(\mathbb{R}^n) \to L^2_{\mathrm{comp}}(\mathbb{R}^n), \quad \text{for } \chi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n), \end{aligned} \tag{2.11}$$

are meromorphic families of operators for $\lambda \in \mathbb{C}$ when n is odd, $\lambda \in \Lambda$ when n is even.

Denote by $R_0(\lambda, x, y)$ the convolution kernel associated to the operator $R_0(\lambda)$, which can be written in terms of the Hankel functions:

$$R_0(\lambda, x, y) = c_n \lambda^{n-2} (\lambda|x-y|)^{-\frac{n-2}{2}} H^{(1)}_{\frac{n}{2}-1}(\lambda|x-y|). \tag{2.12}$$

We recall some well known facts about $R_0(\lambda, x, y)$ as follows, see for instance [DyZw19, §3.1] for a detailed account. When n is odd, (2.12) admits a finite expansion:

$$R_0(\lambda, x, y) = \lambda^{n-2} e^{i\lambda|x-y|} \sum_{j=\frac{n-1}{2}}^{n-2} \frac{c_{n,j}}{(\lambda|x-y|)^j}. \tag{2.13}$$

For $x \neq y$ this form extends meromorphically to $\lambda \in \mathbb{C}$. When n is even, using the relation:

$$R_0(e^{i\ell\pi}\lambda, x, y) - R_0(\lambda, x, y) = c_n \ell (-1)^{\frac{n-2}{2}(\ell+1)} \lambda^{\frac{n-2}{2}} |x-y|^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda|x-y|), \quad (2.14)$$

where $J_d(z)$ is the Bessel function, we see that $R_0(\lambda, x, y)$, $x \neq y$ extends to $\lambda \in \Lambda$. In view of (2.13), for n odd we have the upper bounds

$$|R_0(\lambda, x, y)| \lesssim \begin{cases} |x-y|^{2-n}, & |x-y| \leq |\lambda|^{-1}; \\ e^{-\operatorname{Im}\lambda|x-y|}|\lambda|^{\frac{n-3}{2}}|x-y|^{\frac{1-n}{2}}, & |x-y| \geq |\lambda|^{-1}. \end{cases} \quad (2.15)$$

For n even, $n \neq 2$, the bounds (2.15) hold for $-\pi < \arg\lambda < 2\pi$. This follows from the asymptotics of $H_d^{(1)}(z)$, see also Galkowski–Smith [GaSm14] for more details. Using (2.14) and the aysmptotics of $J_d(z)$ we can then extend (2.15) to any compact subsets of Λ. In the case that $n = 2$, in (2.15) $|x-y|^{2-n}$ is replaced by $-\ln|x-y|$ in the bound for $|x-y| \leq |\lambda|^{-1}$. Now we can conclude from (2.15) that for any fixed (except possible poles) $\lambda \in \mathbb{C}$ when n is odd, $\lambda \in \Lambda$ when n is even,

$$\sup_x \int_{\mathbb{R}^n} e^{-|x|^2-|y|^2}|R_0(\lambda, x, y)|dy, \quad \sup_y \int_{\mathbb{R}^n} e^{-|x|^2-|y|^2}|R_0(\lambda, x, y)|dx < \infty.$$

Using the formula about derivatives of the Hankel functions

$$\frac{d}{dz}H_m^{(1)}(z) = H_{m-1}^{(1)}(z) - \frac{m}{z}H_m^{(1)}(z),$$

we can also conclude from the bounds (2.15) that

$$\sup_x \int_{\mathbb{R}^n} |[\Delta_x, \chi]R_0(\lambda, x, y)|e^{-|y|^2}dy, \quad \sup_y \int_{\mathbb{R}^n} |[\Delta_x, \chi]R_0(\lambda, x, y)|e^{-|y|^2}dx < \infty.$$

Hence (2.11) follows by the Schur test.

3. The deformed operators

In this section we conjugate $\Delta_{\Phi(\mathcal{O})}$ by the pullback Φ^* for any $\Phi \in \mathrm{Diff}(\mathcal{O})$. This will transform the deformed domain $\mathbb{R}^n \setminus \Phi(\mathcal{O})$ to the original one. As a result, the variation is transferred to the coefficients of the newly-defined differential operator.

For $\mathcal{O}$, an obstacle, and $\Phi \in \mathrm{Diff}(\mathcal{O})$ given in (1.5), the pullback Φ^* is a bounded operator from $L^2(\mathbb{R}^n \setminus \Phi(\mathcal{O}))$ to $L^2(\mathbb{R}^n \setminus \mathcal{O})$, which is invertible with the inverse $(\Phi^{-1})^*$. In view of (1.1), the restricted map $\Phi^* : \mathcal{D}(\Delta_{\Phi(\mathcal{O})}) \to \mathcal{D}(\Delta_{\mathcal{O}})$ is also invertible with the inverse $(\Phi^{-1})^*$, since Φ^* preserves the Dirichlet boundary condition. Hence we can define the deformed operator $\Delta_{\mathcal{O}}^{\Phi}$ of $\Delta_{\mathcal{O}}$ associated to the deformation Φ:

$$\Delta_{\mathcal{O}}^{\Phi} := \Phi^* \Delta_{\Phi(\mathcal{O})} (\Phi^{-1})^* : L^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\mathbb{R}^n \setminus \mathcal{O}), \quad \text{with domain } \mathcal{D}(\Delta_{\mathcal{O}}). \quad (3.1)$$

Let $J_\Phi^{jk}(x)$ denote $[D\Phi(x)^{-1}]_{jk}$, by a direct calculation we have

$$\Phi^*\Delta(\Phi^{-1})^* = \sum_{jk\ell} J_\Phi^{\ell j} J_\Phi^{kj} \partial^2_{x_k x_\ell} + \sum_{ijk\ell m} (\partial^2_{x_i x_\ell}\Phi^m) J_\Phi^{km} J_\Phi^{ij} J_\Phi^{\ell j} \partial_{x_k},$$

where $\Phi^m(x)$ is the m-th component of $\Phi(x) = (\Phi^1(x), \cdots, \Phi^n(x))$. Now we define

$$\begin{aligned} V &:= \Delta - \Phi^*\Delta(\Phi^{-1})^* = \sum_{k,\ell} a_{k\ell}(x)\partial^2_{x_k x_\ell} + \sum_k b_k(x)\partial_{x_k}, \\ \text{where} \quad a_{k\ell} &= \delta_{k\ell} - \sum_j J_\Phi^{\ell j} J_\Phi^{kj}, \quad b_k = -\sum_{ij\ell m} (\partial^2_{x_i x_\ell}\Phi^m) J_\Phi^{km} J_\Phi^{ij} J_\Phi^{\ell j}, \end{aligned} \tag{3.2}$$

then by (1.5) we obtain that for all $1 \le k, \ell \le n$,

$$a_{k\ell},\, b_k \in C_c^2(\mathbb{R}^n), \quad \sup_{\mathbb{R}^n\setminus\mathcal{O}} |a_{k\ell}(x)|,\, \sup_{\mathbb{R}^n\setminus\mathcal{O}} |b_k(x)| \le C\|\Phi - \mathrm{id}\,\|_{C^2(\mathbb{R}^n\setminus\mathcal{O})}. \tag{3.3}$$

We note that $-\Delta_\mathcal{O}$ is a self-adjoint black box Hamiltonian, whose resolvent admits a meromorphic continuation by Proposition 2.1. More precisely, for any obstacle $\mathcal{O}$, the resolvent of $-\Delta_\mathcal{O}$ is a meromorphic family of operators from $e^{-|x|^2}L^2(\mathbb{R}^n \setminus \mathcal{O})$ to $\mathcal{D}_1(\mathcal{O})$, where $\mathcal{D}_1(\mathcal{O})$ is the same as (2.8) except that $e^{|x|^2}L^2(\mathbb{R}^n\setminus\mathcal{O})$ replaces $\mathcal{H}_1$ there:

$$\mathcal{D}_1(\mathcal{O}) = \{u \in e^{|x|^2}L^2(\mathbb{R}^n\setminus\mathcal{O}) \cap H^2_{\mathrm{loc}}(\mathbb{R}^n\setminus\mathcal{O}) : u|_{\partial\mathcal{O}} = 0,\ \Delta u \in e^{|x|^2}L^2(\mathbb{R}^n\setminus\mathcal{O})\}. \tag{3.4}$$

Since Φ^* maps $\mathcal{D}_1(\Phi(\mathcal{O}))$ to $\mathcal{D}_1(\mathcal{O})$, $e^{-|x|^2}L^2(\mathbb{R}^n\setminus\Phi(\mathcal{O}))$ to $e^{-|x|^2}L^2(\mathbb{R}^n\setminus\mathcal{O})$ respectively, it follows from (3.1) that the resolvent of $-\Delta_\mathcal{O}^\Phi$ also has a meromorphic continuation given by

$$(-\Delta_\mathcal{O}^\Phi - \lambda^2)^{-1} = \Phi^*(-\Delta_{\Phi(\mathcal{O})} - \lambda^2)^{-1}(\Phi^{-1})^* = \Phi^* R_{\Phi(\mathcal{O})}(\lambda)(\Phi^{-1})^*, \tag{3.5}$$

whose poles, denoted by $\mathrm{Res}(-\Delta_\mathcal{O}^\Phi)$, coincide, with agreement of multiplicities, with the resonances of $\Phi(\mathcal{O})$.

4. Perturbation Theory of Resonances

In this section we adapt the abstract framework introduced by Agmon [Ag98] to study the behavior of resonances under obstacle deformations in which the resonances will be characterized as the eigenvalues of a non-self-adjoint operator.

For a fixed obstacle $\mathcal{O}$, we consider the family of operators $-\Delta_\mathcal{O}^\Phi$ where Φ ranges over $\mathrm{Diff}(\mathcal{O})$. We have shown in §3 that the resolvent $(-\Delta_\mathcal{O}^\Phi - \lambda^2)^{-1}$ admits a meromorphic continuation to $\mathbb{C}$ when n is odd, Λ when n is even, as an operator from $B_0 := e^{-|x|^2}L^2(\mathbb{R}^n\setminus\mathcal{O})$ to $B_1 := e^{|x|^2}L^2(\mathbb{R}^n\setminus\mathcal{O})$. We will replace parameter λ by z with $z = \lambda^2$, and write $\mathcal{R}_\Phi(z) := (-\Delta_\mathcal{O}^\Phi - z)^{-1} : B_0 \to B_1$. We will also write $\mathrm{Res}(\mathcal{O})$, $\mathrm{Res}(-\Delta_\mathcal{O}^\Phi)$ for the image of the sets of resonances under the map $\lambda \mapsto z = \lambda^2$.

We note that $-\Delta_{\mathcal{O}}$ as an operator acting on B_1 is closable. Denote by P_1 the closure of $-\Delta_{\mathcal{O}}$ in B_1, by (3.4) we have

$$P_1 = -\Delta : B_1 \to B_1 \quad \text{with domain } \mathcal{D}_1(\mathcal{O}). \tag{4.1}$$

It then follows from (3.1) and (3.2) that $-\Delta_{\mathcal{O}}^{\Phi}$ is also closable on B_1 with the closure

$$P_1^{\Phi} := \Phi^*(-\Delta)(\Phi^{-1})^* = -\Delta + V : B_1 \to B_1 \quad \text{with domain } \mathcal{D}_1(\mathcal{O}). \tag{4.2}$$

Hence, both $\Delta_{\mathcal{O}}$ and the deformed operator $-\Delta_{\mathcal{O}}^{\Phi}$ satisfy the hypotheses of Agmon's abstract theory [Ag98]: for an operator P on a Banach space $B_0 \subset B \subset B_1$ (we take $B = L^2(\mathbb{R}^n \setminus \mathcal{O})$), (i) the resolvent $(P-z)^{-1}$ admits a meromorphic continuation from D to D_+ where $\sigma_{\text{ess}}(P) \cap D_+ \neq \emptyset$; (ii) P as an operator on B_1 is closable; (iii) Denote by P_1 the closure of P in B_1, the resolvent $(P_1 - z)^{-1}$ exists for some $\lambda \in D$.

To study the perturbation of a resonance $z_0 \in \operatorname{Res}(\mathcal{O})$, $z_0 \neq 0$ by (1.3), let D_+ be a bounded domain containing z_0 such that $\overline{D_+} \subset \mathbb{C} \setminus \{0\}$ when n is odd, $\overline{D_+} \subset \Lambda$ when n is even, with a C^1 boundary Γ, satisfying

$$\text{(i) } \Gamma \cap \operatorname{Res}(\mathcal{O}) = \emptyset; \quad \text{(ii) } D_+ \cap \{z : 0 < \operatorname{Im} z < \pi\} \neq \emptyset. \tag{4.3}$$

Having chosen D_+ we denote by B_Γ, the subspace of B_1 consisting of elements f, admitting a representation of the form:

$$f = g + \int_\Gamma \mathcal{R}(\zeta)\varphi(\zeta)d\zeta, \quad g \in B_0,\ \varphi \in C(\Gamma; B_0), \tag{4.4}$$

where $\mathcal{R}(\zeta) := (-\Delta_{\mathcal{O}} - \zeta)^{-1} : B_0 \to \mathcal{D}_1(\mathcal{O})$ denotes the meromorphically continued resolvent. We recall [Ag98] that B_Γ is a Banach space with the norm

$$\|f\|_{B_\Gamma} := \inf_{g,\varphi}(\|g\|_{B_0} + \|\varphi\|_{C(\Gamma;B_0)}) \tag{4.5}$$

where the infimum is taken over all $g \in B_0$ and $\varphi \in C(\Gamma; B_0)$ which verify (4.4). Then $B_0 \subset B_\Gamma \subset B_1$ are continuous inclusions. Agmon [Ag98] also introduced a linear operator $R_\Gamma(z)$ on B_Γ associated to any $z \in D_+ \setminus \operatorname{Res}(\mathcal{O})$,

$$R_\Gamma(z)f := \mathcal{R}(z)g + \int_\Gamma (\zeta - z)^{-1}(\mathcal{R}(\zeta) - \mathcal{R}(z))\varphi(\zeta)d\zeta, \tag{4.6}$$

where $f \in B_\Gamma$ is given by (4.4). It has been shown that $R_\Gamma(z)$ is a well-defined operator in $\mathcal{L}(B_\Gamma)$ which is actually the resolvent of an operator P_Γ in B_Γ:

$$R_\Gamma(z) = (P_\Gamma - z)^{-1} \quad \text{for} \quad z \in D \setminus \operatorname{Res}(\mathcal{O}), \tag{4.7}$$

where P_Γ is closed linear operator in B_Γ defined as follows:

$$\mathcal{D}(P_\Gamma) = \operatorname{Ran} R_\Gamma(w_0), \quad P_\Gamma u = w_0 u + f \tag{4.8}$$

for $u = R_\Gamma(w_0)f \in \mathcal{D}(P_\Gamma)$, $f \in B_\Gamma$. Here w_0 is a fixed point in $D_+ \cap \{0 < \arg z < \pi\}$. Moreover, P_1 extends P_Γ in the sense that

$$\mathcal{D}(P_\Gamma) \subset \mathcal{D}_1(\mathcal{O}), \quad P_\Gamma u = P_1 u \quad \text{for } u \in \mathcal{D}(P_\Gamma), \tag{4.9}$$

where $\mathcal{D}(P_\Gamma) \subset \mathcal{D}_1(\mathcal{O})$ is continuous if they are equipped with the graph norms:

$$\|u\|_{\mathcal{D}(P_\Gamma)} := \|u\|_{B_\Gamma} + \|P_\Gamma u\|_{B_\Gamma}; \quad \|u\|_{\mathcal{D}_1(\mathcal{O})} := \|u\|_{B_1} + \|\Delta u\|_{B_1}.$$

Agmon [Ag98] proved the following properties that relate P_Γ to the operator $-\Delta_{\mathcal{O}}$:

Proposition 4.1. *P_Γ has a discrete spectrum in D_+, given by $\mathrm{Res}(\mathcal{O}) \cap D_+$. Furthermore, let $z_0 \in \mathrm{Res}(\mathcal{O}) \cap D_+$ be an eigenvalue of P_Γ, $\mathcal{E}_\Gamma(z_0)$ denote the generalized eigenspace of P_Γ at z_0, then*

$$\mathcal{E}_\Gamma(z_0) := \left(\oint_{z_0} (P_\Gamma - \zeta)^{-1} d\zeta\right)(B_\Gamma) = \left(\oint_{z_0} \mathcal{R}(\zeta) d\zeta\right)(B_0), \tag{4.10}$$

where the integral is over a circle containing no other resonance than z_0. In particular, the multiplicity of $z_0 \in \mathrm{Spec}(P_\Gamma)$ satisfies

$$m_\Gamma(z_0) := \dim \mathcal{E}_\Gamma(z_0) = m_{\mathcal{O}}(\lambda_0), \quad \text{with } z_0 = \lambda_0^2. \tag{4.11}$$

Now we consider Agmon's theory for the deformed operators $-\Delta_{\mathcal{O}}^\Phi$. For a fixed domain D_+ with boundary Γ satisfying (4.3), by Proposition 2.1 we assume that

$$\sup_{\zeta \in \Gamma} \|\mathcal{R}(\zeta)\|_{\mathcal{L}(B_0, \mathcal{D}_1(\mathcal{O}))} < C_\Gamma \quad \text{for some constant } C_\Gamma > 0. \tag{4.12}$$

Since V defined in (3.2) can be viewed as an operator in $\mathcal{L}(\mathcal{D}_1(\mathcal{O}), B_0)$ satisfying

$$\|V\|_{\mathcal{L}(\mathcal{D}_1(\mathcal{O}), B_0)} \le C \|\Phi - \mathrm{id}\,\|_{C^2(\mathbb{R}^n \setminus \mathcal{O})}, \tag{4.13}$$

there exists $\delta_\Gamma > 0$ sufficiently small such that for all $\zeta \in \Gamma$,

$$\|\Phi - \mathrm{id}\,\|_{C^2(\mathbb{R}^n \setminus \mathcal{O})} < \delta_\Gamma \implies \|V\mathcal{R}(\zeta)\|_{\mathcal{L}(B_0, B_0)} < 1/2, \tag{4.14}$$

which guarantees that $I + V\mathcal{R}(\zeta) : B_0 \to B_0$ is invertible by a Neumann series argument. Thus we have

$$\mathcal{R}_\Phi(\zeta) := (-\Delta_{\mathcal{O}}^\Phi - \zeta)^{-1} = \mathcal{R}(\zeta)(I + V\mathcal{R}(\zeta))^{-1}, \quad \zeta \in \Gamma, \tag{4.15}$$

which can be justified first for ζ near $\Gamma \cap \{z : 0 < \mathrm{Im}\, z < \pi\}$ and then by meromorphic continuation. In particular, $\Gamma \cap \mathrm{Res}(-\Delta_{\mathcal{O}}^\Phi) = \emptyset$. Hence for the same domain D_+ with boundary Γ, we can define $B_{\Gamma,\Phi}$, $R_{\Gamma,\Phi}$ and $P_{\Gamma,\Phi}$ for the deformed operator $-\Delta_{\mathcal{O}}^\Phi$, as in (4.4), (4.6) and (4.8) with $\mathcal{R}(\zeta)$ replaced by $\mathcal{R}_\Phi(\zeta)$.

Now we explore the relationships between $B_{\Gamma,\Phi}$, $R_{\Gamma,\Phi}$, $P_{\Gamma,\Phi}$ and B_Γ, R_Γ, P_Γ. Assuming that $\|\Phi - \mathrm{id}\,\|_{C^2(\mathbb{R}^n \setminus \mathcal{O})} < \delta_\Gamma$, by (4.15) we have for any $f \in B_\Gamma$,

$$f = g + \int_\Gamma \mathcal{R}(\zeta)\varphi(\zeta) d\zeta = g + \int_\Gamma \mathcal{R}_\Phi(\zeta)(I + V\mathcal{R}(\zeta))\varphi(\zeta) d\zeta.$$

Since $(I + V\mathcal{R}(\zeta))\varphi(\zeta) \in C(\Gamma; B_0)$, $f \in B_{\Gamma,\Phi}$ thus we have $B_\Gamma \subset B_{\Gamma,\Phi}$. Furthermore, (4.14) implies that

$$\|g\|_{B_0} + \|(I + V\mathcal{R}(\zeta))\varphi(\zeta)\|_{C(\Gamma; B_0)} \le \frac{3}{2}(\|g\|_{B_0} + \|\varphi\|_{C(\Gamma; B_0)}),$$

by taking the infimum as in (4.5), we obtain that $\|f\|_{B_{\Gamma,\Phi}} \leq 3/2\,\|f\|_{B_\Gamma}$. Similarly, for $f \in B_{\Gamma,\Phi}$ we have

$$f = g + \int_\Gamma \mathcal{R}_\Phi(\zeta)\varphi(\zeta)d\zeta = g + \int_\Gamma \mathcal{R}(\zeta)(I + V\mathcal{R}(\zeta))^{-1}\varphi(\zeta)d\zeta \in B_\Gamma,$$

and again by (4.14) we can deduce that $\|f\|_{B_\Gamma} \leq 2\,\|f\|_{B_{\Gamma,\Phi}}$. Therefore,

$$B_{\Gamma,\Phi} = B_\Gamma,\ \ \|\cdot\|_{B_{\Gamma,\Phi}} \text{ and } \|\cdot\|_{B_\Gamma} \text{ are equivalent}, \quad \text{if } \|\Phi - \mathrm{id}\,\|_{C^2(\mathbb{R}^n\setminus\mathcal{O})} < \delta_\Gamma. \tag{4.16}$$

Henceforth, we identify $B_{\Gamma,\Phi}$ with B_Γ. Suppose that $f = g + \int_\Gamma \mathcal{R}_\Phi(\zeta)\varphi(\zeta)d\zeta \in B_\Gamma$, then for w_0 chosen in (4.8), in view of (4.6) and (4.15) we have

$$\begin{aligned} R_{\Gamma,\Phi}(w_0)f &= \mathcal{R}_\Phi(w_0)g + \int_\Gamma (\zeta - w_0)^{-1}(\mathcal{R}_\Phi(\zeta) - \mathcal{R}_\Phi(w_0))\varphi(\zeta)d\zeta \\ &= \mathcal{R}(w_0)g_1 + \int_\Gamma (\zeta - w_0)^{-1}(\mathcal{R}(\zeta) - \mathcal{R}(w_0))\varphi_1(\zeta)d\zeta, \end{aligned}$$

where $\varphi_1(\zeta) := (I + V\mathcal{R}(\zeta))^{-1}\varphi(\zeta) \in C(\Gamma; B_0)$ and

$$g_1 := (I + V\mathcal{R}(w_0))^{-1}g + \int_\Gamma \frac{(I + V\mathcal{R}(\zeta))^{-1} - (I + V\mathcal{R}(w_0))^{-1}}{\zeta - w_0}\varphi(\zeta)d\zeta \ \in B_0.$$

Thus $R_{\Gamma,\Phi}(w_0)f = R_\Gamma(w_0)f_1$ for $f_1 := g_1 + \int_\Gamma \mathcal{R}(\zeta)\varphi_1(\zeta)d\zeta \in B_\Gamma$, which implies that $\operatorname{Ran} R_{\Gamma,\Phi}(w_0) \subset \operatorname{Ran} R_\Gamma(w_0)$. We can also derive that $\operatorname{Ran} R_\Gamma(w_0) \subset \operatorname{Ran} R_{\Gamma,\Phi}(w_0)$ by similar arguments. Therefore, recalling (4.8) we obtain that

$$\mathcal{D}(P_{\Gamma,\Phi}) := \operatorname{Ran} R_{\Gamma,\Phi}(w_0) = \operatorname{Ran} R_\Gamma(w_0) = \mathcal{D}(P_\Gamma).$$

We recall [Ag98] that P_1^Φ extends $P_{\Gamma,\Phi}$ as in (4.9), then for any $u \in \mathcal{D}(P_\Gamma)$, (4.2) and (4.9) imply that

$$P_{\Gamma,\Phi}u = P_1^\Phi u = P_1 u + Vu = P_\Gamma u + Vu \tag{4.17}$$

Hence $P_{\Gamma,\Phi}$ and P_Γ are related as follows

$$P_{\Gamma,\Phi} = P_\Gamma + V : B_\Gamma \to B_\Gamma \quad \text{with domain } \mathcal{D}(P_\Gamma). \tag{4.18}$$

Now we substitute P_Γ by $P_{\Gamma,\Phi}$ in Proposition 4.1 and recall (3.5) to conclude:

Proposition 4.2. *Let D_+ with boundary Γ be chosen as in (4.3) and suppose that $\Phi \in \mathrm{Diff}(\mathcal{O})$ satisfies $\|\Phi - \mathrm{id}\,\|_{C^2(\mathbb{R}^n\setminus\mathcal{O})} < \delta_\Gamma$ for some $\delta_\Gamma > 0$ in (4.14), then $P_{\Gamma,\Phi}$ has a discrete spectrum in D_+, given by $\mathrm{Res}(\Phi(\mathcal{O})) \cap D_+$.*

Furthermore, let $z \in \mathrm{Res}(\Phi(\mathcal{O})) \cap D_+$ be an eigenvalue of $P_{\Gamma,\Phi}$, denote by $\mathcal{E}_{\Gamma,\Phi}(z)$ the generalized eigenspace of $P_{\Gamma,\Phi}$ at z, then

$$\mathcal{E}_{\Gamma,\Phi}(z) := \left(\oint_z (P_{\Gamma,\Phi} - \zeta)^{-1}d\zeta\right)(B_\Gamma) = \left(\oint_z \mathcal{R}_\Phi(\zeta)d\zeta\right)(B_0) \tag{4.19}$$

where the integral is over a circle containing no other resonance than z. In particular, the multiplicity of $z \in \operatorname{Spec} P_{\Gamma,\Phi}$ satisfies

$$m_{\Gamma,\Phi}(z) := \dim \mathcal{E}_{\Gamma,\Phi}(z) = m_{\Phi(\mathcal{O})}(\lambda), \quad \textit{with } z = \lambda^2. \tag{4.20}$$

5. Generic Simplicity of Resonances

We will follow the strategy of [KlZw95] and [BoPe02] in the case of potential perturbations to prove our theorem stated in §1. However we have to resolve the additional difficulties produced by boundary perturbations using the results obtained in §3 and §4. For simplicity we identify $\mathbb{C}\setminus\{0\}$ with $\{\lambda\in\Lambda : -\pi\le\arg\lambda<\pi\}$ when n is odd. Let X be a class of obstacles given by (1.6), for any $\theta>0$ and $r>1$, we define

$$\begin{aligned} S_\theta^r &:= \{\lambda\in\Lambda : -\theta<\arg\lambda<\theta,\ 1/r<|\lambda|<r\},\\ E_\theta^r &:= \{\mathcal{O}\in X : m_{\mathcal{O}}(\lambda)\le 1, \quad \forall\,\lambda\in S_\theta^r\}. \end{aligned} \tag{5.1}$$

It suffices to show that for each θ and r, E_θ^r is open and dense, since we can then obtain the generic set $\mathcal{X}$ by taking

$$\mathcal{X} := \bigcap_{m=1}^{\infty}\bigcap_{N=1}^{\infty} E_{m\pi}^N \text{ when } n \text{ is even;}\quad \mathcal{X} := \bigcap_{N=1}^{\infty} E_{2\pi}^N \text{ when } n \text{ is odd.}$$

We proceed the proof of our theorem in steps:

Proof. 1. For $\mathcal{O}\in X$ we will write $\operatorname{Res}(\mathcal{O})$ for the image of resonances under the map $\lambda\mapsto z=\lambda^2$, and for any z the multiplicity is given by $m_{\mathcal{O}}(z):=m_{\mathcal{O}}(\lambda)$ provided $z=\lambda^2$. Then $E_\theta^r=\{\mathcal{O}\in X : m_{\mathcal{O}}(z)\le 1,\ \forall\, z\in S_{2\theta}^{r^2}\}$. Suppose that there is exactly one resonance z_0 in $B(z_0,2\delta)\subset S_{2\theta}^{r^2}$, where $B(z_0,r):=\{z_0+w : w\in\mathbb{C},\ |w|<r\}$. For $\Omega:=B(z_0,\delta)$ we then define

$$\Pi_{\mathcal{O}}(\Omega) := -\frac{1}{2\pi i}\int_{\partial\Omega}(-\Delta_{\mathcal{O}}-\zeta)^{-1}d\zeta, \quad m_{\mathcal{O}}(\Omega):=\operatorname{rank}\Pi_{\mathcal{O}}(\Omega). \tag{5.2}$$

Now we choose a bounded domain D_+ containing $B(z_0,2\delta)$ with boundary Γ satisfying (4.3). We also assume that $\overline{D_+}\subset S_{2\theta}^{r^2}$. By Proposition 4.1, elements in $\operatorname{Res}(\mathcal{O})$ coincide with the eigenvalues of P_Γ in D_+. In view of (4.11), we have the relationship:

$$\Pi_\Gamma(\Omega) := -\frac{1}{2\pi i}\int_{\partial\Omega}(P_\Gamma-\zeta)^{-1}d\zeta, \quad m_\Gamma(\Omega):=\operatorname{rank}\Pi_\Gamma(\Omega)=m_{\mathcal{O}}(\Omega). \tag{5.3}$$

Let $\mathcal{U}_\varepsilon(\mathcal{O})$ be a set of deformations defined for small $\varepsilon>0$,

$$\mathcal{U}_\varepsilon(\mathcal{O}) := \{\Phi\in\operatorname{Diff}(\mathcal{O}) : \|\Phi-\operatorname{id}\|_{C^2(\mathbb{R}^n\setminus\mathcal{O})}<\varepsilon\}.$$

Assuming that $\varepsilon<\delta_\Gamma$ for constant δ_Γ given in (4.14), then for every $\Phi\in\mathcal{U}_\varepsilon(\mathcal{O})$ Proposition 4.2 implies that

$$\Pi_{\Gamma,\Phi}(\Omega) := -\frac{1}{2\pi i}\int_{\partial\Omega}(P_{\Gamma,\Phi}-\zeta)^{-1}d\zeta, \quad m_{\Gamma,\Phi}(\Omega):=\operatorname{rank}\Pi_{\Gamma,\Phi}(\Omega)=m_{\Phi(\mathcal{O})}(\Omega). \tag{5.4}$$

We recall (4.18) that $P_{\Gamma,\Phi}=P_\Gamma+V$ with V defined in (3.2), then by (4.13) we obtain that if ε is sufficiently small, then for $\zeta\in\partial\Omega$ and $\Phi\in\mathcal{U}_\varepsilon(\mathcal{O})$,

$$(P_{\Gamma,\Phi}-\zeta)^{-1} = (P_\Gamma-\zeta)^{-1}(I+V(P_\Gamma-\zeta)^{-1})^{-1}$$

and $\sup_{\zeta\in\partial\Omega}\|(P_{\Gamma,\Phi}-\zeta)^{-1}-(P_\Gamma-\zeta)^{-1}\|_{B_\Gamma\to B_\Gamma}<C(\Omega)\varepsilon$. Then we can derive that $\Pi_\Gamma(\Omega)$ and $\Pi_{\Gamma,\Phi}(\Omega)$ have the same rank for any $\Phi\in\mathcal{U}_\varepsilon(\mathcal{O})$ if ε is sufficiently small. We restate this as follows:

$$m_{\Phi(\mathcal{O})}(\Omega) \text{ is constant for } \Phi\in\mathcal{U}_\varepsilon(\mathcal{O}) \text{ if } \varepsilon \text{ is sufficiently small.} \tag{5.5}$$

Hence $\mathcal{O}\in E_\theta^r$ implies that $\{\Phi(\mathcal{O}):\Phi\in\mathcal{U}_\varepsilon(\mathcal{O})\}\subset E_\theta^r$ for some ε sufficiently small, in other words, E_θ^r is open.

2. It remains to show that E_θ^r is dense, which follows from the following:

$$\forall\,\mathcal{O}\in X \text{ and } \varepsilon>0,\quad \exists\,\Phi\in\mathcal{U}_\varepsilon(\mathcal{O}) \text{ such that } \Phi(\mathcal{O})\in E_\theta^r. \tag{5.6}$$

Since the number of resonances for the obstacle $\mathcal{O}$ in S_θ^r is finite, it is enough to prove a local statement as it can be applied successively to obtain (5.6) (once a resonance is simple it stays simple under small deformations due to (5.5)). We will define Ω for any given $\mathcal{O}$ and $z_0\in\mathrm{Res}(\mathcal{O})$ as in Step 1, thus to obtain (5.6) it suffices to prove that for

$$\forall\,\mathcal{O}\in X,\ z_0\in\mathrm{Res}(\mathcal{O}) \text{ and } \varepsilon>0,\quad \exists\,\Phi\in\mathcal{U}_\varepsilon(\mathcal{O}) \ \text{ s.t. } m_{\Phi(\mathcal{O})}(z)\le 1,\ \forall\, z\in\Omega. \tag{5.7}$$

To establish (5.7) we proceed by induction. We note that for each $\mathcal{O}\in X$, $z_0\in\mathrm{Res}(\mathcal{O})$, one of the following cases has to occur:

$$\forall\,\varepsilon>0,\quad \exists\,\Phi\in\mathcal{U}_\varepsilon(\mathcal{O}) \ \text{ s.t. } 1\le m_{\Phi(\mathcal{O})}(z)<m_{\Phi(\mathcal{O})}(\Omega),\quad \forall\, z\in\Omega, \tag{5.8}$$

or

$$\exists\,\varepsilon>0,\ \text{s.t.}\quad \forall\,\Phi\in\mathcal{U}_\varepsilon(\mathcal{O}),\ \exists z=z(\Phi)\in\Omega,\ m_{\Phi(\mathcal{O})}(z)=m_{\Phi(\mathcal{O})}(\Omega)>1. \tag{5.9}$$

The first possibility means that by applying an arbitrarily small deformation Φ to $\mathcal{O}$ we can obtain at least two distinct resonances in $\mathrm{Res}(\Phi(\mathcal{O}))$ in Ω. The second possibility means that under any small deformations the maximal multiplicity persists.

3. Assuming (5.8) we can prove (5.7) by induction on $m_{\mathcal{O}}(z_0)$. If $m_{\mathcal{O}}(z_0)=1$ there is nothing to prove. Assuming that we proved (5.7) in the case $m_{\mathcal{O}}(z_0)<M$, we now assume that $m_{\mathcal{O}}(z_0)=M$. We note that for any $\Phi_1\in\mathrm{Diff}(\mathcal{O})$ and $\Phi_2\in\mathrm{Diff}(\Phi_1(\mathcal{O}))$, there exists $C=C(n)$ such that

In view of (5.8) we can find $\Phi_0\in\mathrm{Diff}(\mathcal{O})$ with $\|\Phi_0-\mathrm{id}\,\|_{C^2(\mathbb{R}^n\setminus\mathcal{O})}<\varepsilon/(2C)^M$ such that $m_{\Phi_0(\mathcal{O})}(\Omega)=m_{\mathcal{O}}(\Omega)$ (using (5.5)) and all resonances in Ω, $z_1,\cdots,z_k$, satisfy $m_{\Phi_0(\mathcal{O})}(z_j)<M$. We now find r_j such that

$$B(z_j,2r_j)\subset\Omega,\ \{z_j\}=B(z_j,2r_j)\cap\mathrm{Res}(\Phi_0(\mathcal{O})),\quad B(z_j,2r_j)\cap B(z_k,2r_k)=\emptyset.$$

We put $\Omega_j:=B(z_j,r_j)$ and apply (5.7) successively to $\Phi_{j-1}\circ\cdots\circ\Phi_0(\mathcal{O})$, $j=1,\cdots,k$, in Ω_j with $\|\Phi_j-\mathrm{id}\,\|_{C^2(\mathbb{R}^n\setminus\Phi_{j-1}\circ\cdots\circ\Phi_0(\mathcal{O}))}<\varepsilon/(2C)^{k+1-j}$ (by (5.5) we can assume that Φ_j is sufficiently close to id such that resonances in $\Omega_0,\cdots,\Omega_{j-1}$ that are already simple stay simple while multiplicities of $\Omega_{j+1}, \cdots, \Omega_k$ are invariant). Then we obtain the desired $\Phi = \Phi_k \circ \cdots \circ \Phi_0 \in \mathcal{U}_\varepsilon(\mathcal{O})$ since

$$\|\Phi_k \circ \cdots \circ \Phi_0 - \mathrm{id}\,\|_{C^2(\mathbb{R}^n \setminus \mathcal{O})} < \sum_{j=1}^{k} C^{k+1-j} \frac{\varepsilon}{(2C)^{k+1-j}} + C^k \frac{\varepsilon}{(2C)^M} \leq \varepsilon.$$

4. It remains to show that (5.9) is impossible. For that, we shall argue by contradiction, assume that $m_{\mathcal{O}}(\Omega) = M$ and that (5.9) holds. Suppose that D_+ and Γ are chosen as in Step 1. Using (5.3) and (5.4) we obtain an equivalent statement to (5.9):

$$\exists\, \varepsilon > 0, \text{ s.t. } \quad \forall\, \Phi \in \mathcal{U}_\varepsilon(\mathcal{O}),\ \exists z = z(\Phi) \in \Omega,\ m_{\Gamma,\Phi}(z) = m_{\Gamma,\Phi}(\Omega) > 1. \tag{5.10}$$

For $\Phi \in \mathcal{U}_\varepsilon(\mathcal{O})$, we define

$$k(\Phi) := \min\{k : (P_{\Gamma,\Phi} - z(\Phi))^k \Pi_{\Gamma,\Phi}(\Omega) = 0\},$$

then $1 \leq k(\Phi) \leq M$. It follows from (4.18) and (3.2) that if $\|\Phi_j - \Phi\|_{C^{2M}} \to 0$ and $(P_{\Gamma,\Phi_j} - z(\Phi_j))^k \Pi_{\Gamma,\Phi_j}(\Omega) = 0$, then $(P_{\Gamma,\Phi} - z(\Phi))^k \Pi_{\Gamma,\Phi}(\Omega) = 0$. We now define

$$k_0 := \max\{k(\Phi) : \Phi \in \mathcal{U}_{\varepsilon/2}(\mathcal{O})\},$$

and assume that the maximum is attained at $\tilde{\Phi} \in \mathcal{U}_{\varepsilon/2}(\mathcal{O})$ i.e. $k(\tilde{\Phi}) = k_0$, then there exists $\delta > 0$ such that $\|\Phi - \tilde{\Phi}\|_{C^{2M}} < \delta \Rightarrow k(\Phi) = k_0$. Henceforth, we can replace our original obstacle $\mathcal{O}$ by $\tilde{\Phi}(\mathcal{O})$, decrease ε and then assume by (5.10) that

$$\begin{gathered} (P_{\Gamma,\Phi} - z(\Phi))^{k_0} \Pi_{\Gamma,\Phi}(\Omega) = 0, \quad (P_{\Gamma,\Phi} - z(\Phi))^{k_0 - 1} \Pi_{\Gamma,\Phi}(\Omega) \neq 0, \\ m_{\Gamma,\Phi}(z(\Phi)) = \operatorname{rank} \Pi_{\Gamma,\Phi}(\Omega) = M > 1, \quad \forall\, \Phi \in \mathrm{Diff}(\mathcal{O}),\ \|\Phi - \mathrm{id}\,\|_{C^{2M}} < \varepsilon. \end{gathered} \tag{5.11}$$

5. Before proving that (5.11) is impossible we introduce a family of deformations in $\mathrm{Diff}(\mathcal{O})$ acting near a fixed point on $\partial\mathcal{O}$. For any fixed $x_0 \in \partial\mathcal{O}$ and some $h_0 > 0$ small we can choose a family of functions $\chi_h \in \mathcal{C}^\infty(\partial\mathcal{O}; [0,\infty))$ depending continuously in $h \in (0, h_0]$ with

$$\int_{\partial\mathcal{O}} \chi_h(x) dS(x) = 1, \quad \operatorname{supp} \chi_h \subset B_{\partial\mathcal{O}}(x_0, h), \quad \forall\, h \in (0, h_0], \tag{5.12}$$

where $B_{\partial\mathcal{O}}(x_0, h)$ is a geodesic ball on $\partial\mathcal{O}$ with center x_0 and radius h. For each $h \in (0, h_0]$, we construct a smooth vector field $V_h \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n; \mathbb{R}^n)$ with some small constant $\delta_h = \mathcal{O}(h^{2M+n-1})$ such that

$$\begin{gathered} V_h(x) = \delta_h \chi_h(x) \nu(x),\ \forall x \in \partial\mathcal{O}, \quad \|V_h\|_{C^{2M}(\mathbb{R}^n;\mathbb{R}^n)} < \varepsilon/2, \\ \operatorname{supp} V_h \subset B_{\mathbb{R}^n}(x_0, Ch) \text{ for some } C > 0, \end{gathered} \tag{5.13}$$

where $\nu(x)$ is the normal vector at $x \in \partial\mathcal{O}$ pointing to the interior of $\mathcal{O}$. Let $\varphi_h^t : \mathbb{R}^n \to \mathbb{R}^n$ be the flow generated by the vector field V_h. It follows from (5.13) that for every $h \in (0, h_0]$ there exists $t_0 > 0$ such that

$$\varphi_h^t \in \mathrm{Diff}(\mathcal{O}), \quad \|\varphi_h^t - \mathrm{id}\,\|_{C^{2M}} < \varepsilon, \quad \forall\, t \in (-t_0, t_0).$$

6. To show that (5.11) is impossible we first assume that $k_0 > 1$. We recall (4.10) that $\Pi_\Gamma(\Omega)(B_\Gamma) = \Pi_\Gamma(\Omega)(B_0)$, let $\mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}}) := \{f \in \mathcal{C}_c^\infty(\mathbb{R}^n) : \operatorname{supp} f \subset \mathbb{R}^n \setminus \overline{\mathcal{O}}\}$ then $\operatorname{Ran} \Pi_\Gamma(\Omega) = \Pi_\Gamma(\Omega)(\mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}}))$ since $\Pi_\Gamma(\Omega)$ is finite rank and $\mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}})$ is dense in B_0. Thus by (5.11) we can find $w \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}})$ such that

$$u := (P_\Gamma - z_0)^{k_0 - 1}\Pi_\Gamma(\Omega)w \neq 0, \quad \text{here } z_0 = z(\mathrm{id}). \tag{5.14}$$

For any fixed $x_0 \in \partial\mathcal{O}$ and $h \in (0, h_0]$, we take $\Phi_t := \varphi_h^t$, $t \in (-t_0, t_0)$ and define

$$u(t) := ({\Phi_t}^{-1})^* v(t), \quad v(t) := (P_{\Gamma,\Phi_t} - z(t))^{k_0-1}\Pi_{\Gamma,\Phi_t}(\Omega)w, \; z(t) := z(\Phi_t). \tag{5.15}$$

Using (4.2) and (4.17), $(P_{\Gamma,\Phi_t} - z(t))v(t) = 0$ implies that

$$(-\Delta - z(t))u(t) = 0 \quad \text{on } \mathbb{R}^n \setminus \Phi_t(\mathcal{O}), \tag{5.16}$$

in the sense of L^2_{loc} functions.

It follows from (4.18) and (3.2) that Π_{Γ,Φ_t} and $u(t)$ depend smoothly on t. By constructing a Grushin problem of $P_{\Gamma,\Phi_t} - z$ for z near z_0 and t near 0, we can obtain a function $f(t,z)$ analytic in z, jointly smooth in t and z, and $f(t,z) \not\equiv 0$ for each fixed t, such that all eigenvalues of P_{Γ,Φ_t} near z_0 coincide with the zeros of $z \mapsto f(t,z)$, for a detailed account see for example [DyZw19, Appendix C]. As t varies near 0, the zeros of $z \mapsto f(t,z)$ do not split (all equal $z(t)$) due to (5.11) and (5.15), thus $z(t)$ depends smoothly on t.

Since $\Phi_t(\mathcal{O}) \subset \mathcal{O}$ for $t \geq 0$, we can restrict (5.16) to the region $\mathbb{R}^n \setminus \mathcal{O}$ then differentiate the equation in t, by taking $t = 0$, we obtain that

$$(-\Delta - z_0)\partial_t u(0,x) = z'(0)u(x) \quad \text{on } \mathbb{R}^n \setminus \mathcal{O}. \tag{5.17}$$

We recall (5.15) that $u(t,x) = v(t, \Phi_{-t}(x))$, using $u(0,x) = v(0,x) = u(x)$ and the flow equation we have

$$\partial_t u(0,x) = \partial_t v(t, \Phi_{-t}(x))|_{t=0} = \partial_t v(0,x) - \partial_x u \cdot V_h(x).$$

In view of (4.10) and (5.14), $u \in \mathcal{E}_\Gamma(z_0)$ is a resonant state of $-\Delta_\mathcal{O}$ at z_0, thus we recall [DyZw19, Theorem 4.7] that $u \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \mathcal{O})$. Then by (5.13) we conclude that

$$\begin{gathered} (-\Delta - z_0)(\partial_t v(0,x) - f) = z'(0)u(x) \quad \text{on } \mathbb{R}^n \setminus \mathcal{O}, \\ f := \partial_x u \cdot V_h(x) \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \mathcal{O}), \quad f|_{\partial\mathcal{O}} = \delta_h \chi_h \partial_\nu u. \end{gathered} \tag{5.18}$$

It follows from $v(t,x) \in \mathcal{D}(P_\Gamma)$, $t \in (-t_0, t_0)$ that $\partial_t v(0,x) \in D(P_\Gamma)$, thus the first equation in (5.18) reduces to

$$(P_\Gamma - z_0)\partial_t v(0,x) = (-\Delta - z_0)f + z'(0)u \quad \text{on } \mathbb{R}^n \setminus \mathcal{O}. \tag{5.19}$$

We introduce the bilinear form on $B_0 \times B_1$ (no complex conjugation),

$$\langle u, v \rangle := \int_{\mathbb{R}^n \setminus \mathcal{O}} uv \, dx, \quad u \in B_1, \; v \in B_0.$$

We now apply the projection Π_Γ (omitting Ω in $\Pi_\bullet(\Omega)$) to both sides of (5.19), pair with $(P_\Gamma - z_0)^{k_0-1}w \in B_0$ (since $w \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}})$), use the fact that $(P_\Gamma - z_0)\Pi_\Gamma g = \Pi_\Gamma(P_\Gamma - z_0)g$, $\forall\, g \in \mathcal{D}(P_\Gamma)$ to obtain that

$$\begin{aligned}&\langle (P_\Gamma - z_0)\Pi_\Gamma \partial_t v(0,x), (P_\Gamma - z_0)^{k_0-1}w\rangle \\ &= \langle \Pi_\Gamma(-\Delta - z_0)f, (P_\Gamma - z_0)^{k_0-1}w\rangle + z'(0)\langle u, (P_\Gamma - z_0)^{k_0-1}w\rangle.\end{aligned}$$

By Green's formula, $\langle P_\Gamma g_1, g_2\rangle = \langle g_1, P_\Gamma g_2\rangle$ for any $g_1 \in \mathcal{D}(P_\Gamma)$, $g_2 \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}})$. It then follows from (5.11) and (5.14) that

$$\langle (P_\Gamma - z_0)\Pi_\Gamma \partial_t v(0,x), (P_\Gamma - z_0)^{k_0-1}w\rangle = \langle (P_\Gamma - z_0)^{k_0}\Pi_\Gamma \partial_t v(0,x), w\rangle = 0,$$

and that

$$\langle u, (P_\Gamma - z_0)^{k_0-1}w\rangle = \langle (P_\Gamma - z_0)u, (P_\Gamma - z_0)^{k_0-2}w\rangle = 0.$$

Since $\langle \Pi_\Gamma f_1, f_2\rangle = \langle \Pi_\Gamma f_2, f_1\rangle$ for any $f_1, f_2 \in B_0$, we conclude that

$$\begin{aligned}0 &= \langle \Pi_\Gamma(-\Delta - z_0)f, (P_\Gamma - z_0)^{k_0-1}w\rangle \\ &= \langle (-\Delta - z_0)f, (P_\Gamma - z_0)^{k_0-1}\Pi_\Gamma w\rangle = \langle (-\Delta - z_0)f, u\rangle.\end{aligned}$$

Now we apply Green's formula and recall (5.18) to obtain that

$$0 = \int_{\partial\mathcal{O}} f\,\partial_\nu u\, dS = \int_{\partial\mathcal{O}} \delta_h \chi_h (\partial_\nu u)^2 dS \implies \int_{\partial\mathcal{O}} \chi_h (\partial_\nu u)^2 dS = 0.$$

Since the above equation holds for any $h \in (0, h_0]$, sending h to $0+$, by (5.12) we can derive that $\partial_\nu u(x_0) = 0$. We note that $x_0 \in \partial\mathcal{O}$ can be chosen arbitrarily, thus $\partial_\nu u|_{\partial\mathcal{O}} \equiv 0$. However, it follows from (5.11) and (5.14) that $u \in \mathcal{D}_1(\mathcal{O})$ satisfying $(-\Delta - z_0)u = 0$ on $\mathbb{R}^n \setminus \mathcal{O}$. Extending u into $\mathcal{O}$ by $u|_{\mathcal{O}} = 0$, it then follows from (5.16) and the boundary values $u|_{\partial\mathcal{O}} = 0$, $\partial_\nu u|_{\partial\mathcal{O}} = 0$ that $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a weak solution of $(-\Delta - z_0)u = 0$ on $\mathbb{R}^n$. The unique continuation property of second order elliptic differential equations shows that $u \equiv 0$, which contradicts (5.14).

7. It remains to consider the case $k_0 = 1$ in (5.11). Let $\{w_j\}_{j=1}^M$ be a set of vectors in $\mathcal{C}_c^\infty(\mathbb{R}^n \setminus \overline{\mathcal{O}})$ such that $\{\Pi_\Gamma w_j\}_{j=1}^M$ is a basis for $\operatorname{Ran}\Pi_\Gamma$. Since Π_Γ is symmetric with respect to the bilinear form $\langle \cdot,\cdot\rangle$ on $B_0 \times B_0$, the matrix A, $A_{ij} := \langle \Pi_\Gamma w_i, w_j\rangle$ is a complex symmetric matrix. To see A is nondegenerate, we suppose that

$$\exists\, x \in \mathbb{C}^M, \quad \langle \Pi_\Gamma w_i, \sum_j x_j w_j\rangle = 0,\ i = 1, \cdots, M.$$

Since $\{\Pi_\Gamma w_i\}_{i=1}^M$ spans $\operatorname{Ran}\Pi_\Gamma$, we have $\langle \Pi_\Gamma w, \sum x_j w_j\rangle = 0$ for all $w \in B_0$, which implies that $\langle \sum x_j \Pi_\Gamma w_j, w\rangle = 0$, $\forall\, w \in B_0$. Hence $\sum x_j \Pi_\Gamma w_j = 0 \Rightarrow x = 0$. We apply the Takagi factorization the the matrix A to obtain that

$$A = U^T \operatorname{Diag}(r_1, \cdots, r_M)U, \text{ where } U \text{ is unitary, } r_j^2 \text{ are the eigenvalues of } AA^*.$$

Using the nondegeneracy of A, we can find B nondegenerate such that $A = B^T B$. We transform $\{w_j\}_{j=1}^M$ by the matrix B and put $u_j := \Pi_\Gamma w_j$, it now follows that

$$\operatorname{Ran}\Pi_\Gamma = \operatorname{span}\{u_j\}_{j=1}^M, \quad \langle u_j, w_i\rangle = \delta_{ij}.$$

For any fixed $x_0 \in \partial\mathcal{O}$ and $h \in (0, h_0]$, we define the evolution of each u_j as in (5.15):

$$u_j(t) := (\Phi_t^{-1})^* v_j(t), \quad v_j(t) := \Pi_{\Gamma,\Phi_t}(\Omega) w_j, \;\; z(t) := z(\Phi_t). \tag{5.20}$$

We note that (5.19) still holds with $\partial_t v(0,x), u, f$ replaced by $\partial_t v_j(0,x), u_j$ and f_j defined as in (5.18). The same arguments as in Step 6 show that

$$\langle (P_\Gamma - z_0)\Pi_\Gamma v_j'(0), w_i\rangle = \langle \Pi_\Gamma(-\Delta - z_0) f_j, w_i\rangle + z'(0)\langle u_j, w_i\rangle.$$

Since $(P_\Gamma - z_0)\Pi_\Gamma = 0$ by (5.11) with $k_0 = 1$, it then follows that

$$\langle (-\Delta - z_0) f_j, u_i\rangle = -z'(0)\delta_{ij}.$$

We apply Green's formula with boundary value of f_j like (5.18) to obtain that

$$-z'(0)\delta_{ij} = \langle (-\Delta - z_0) u_i, f_j\rangle + \int_{\partial\mathcal{O}} f_j \partial_\nu u_i \, dS = -\delta_h \int_{\partial\mathcal{O}} \chi_h (\partial_\nu u_i)(\partial_\nu u_j)\, dS.$$

Hence for any $x_0 \in \partial\mathcal{O}$ and $h \in (0, h_0]$ we have

$$\int_{\partial\mathcal{O}} \chi_h (\partial_\nu u_1)^2 dS = \int_{\partial\mathcal{O}} \chi_h (\partial_\nu u_2)^2 dS, \quad \int_{\partial\mathcal{O}} \chi_h \partial_\nu u_1 \partial_\nu u_2 \, dS = 0.$$

Sending $h \to 0+$, it follows from (5.12) that

$$(\partial_\nu u_1(x_0))^2 = (\partial_\nu u_2(x_0))^2, \quad \partial_\nu u_1(x_0)\partial_\nu u_2(x_0) = 0,$$

thus $\partial_\nu u_1(x_0) = \partial_\nu u_2(x_0) = 0$. Since $x_0 \in \partial\mathcal{O}$ is arbitrary, $\partial_\nu u_1 \equiv 0$. Hence the same arguments as in the end of Step 6 show that $u_1 \equiv 0$, which gives a contradiction. $\square$

References

[Ag98] S. Agmon, *A perturbation theory of resonances,* Communications on Pure and Applied Mathematics **51**(1998), 1255–1309.

[BoPe02] D. Borthwick and P. Perry, *Scattering poles for asymptotically hyperbolic manifolds,* Transactions of the American Mathematical Society **354**(2002), 1215–1231.

[Ch17] T. J. Christiansen, *Lower bounds for resonance counting functions for obstacle scattering in even dimensions,* American Journal of Mathematics **139**(2017), 617–640.

[DyZw19] S. Dyatlov and M. Zworski, *Mathematical theory of scattering resonances,* Graduate Studies in Mathematics **200**, AMS 2019.

[GaSm14] J. Galkowski and H. Smith, *Restriction bounds for the free resolvent and resonances in lossy scattering,* International Mathematics Research Notices **2015**(2015), 7473–7509.

[Ha08] J. Hadamard, *Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées,* Imprimerie nationale **33**, 1908.

[He05] D. Henry, *Perturbation of the boundary in boundary-value problems of partial differential equations,* Cambridge University Press **200**, 2005.

[KlZw95] F. Klopp and M. Zworski, *Generic simplicity of resonances,* Helv. Phys. Acta **68**(1995), 531–538.

[Pe04] M. Pereira, *Generic simplicity of eigenvalues for a Dirichlet problem of the bilaplacian operator,* Electron. J. Differential Equations **2004**(2004), 1–21.

[Uh72] K. Uhlenbeck, *Eigenfunctions of Laplace operators,* Bulletin of the American Mathematical Society **78**(1972), 1073–1076.

[Uh76] K. Uhlenbeck, *Generic properties of eigenfunctions,* American Journal of Mathematics **98**(1976), 1059–1078.

Email address: `xiong@math.berkeley.edu`

Department of Mathematics, University of California, Berkeley, CA 94720, USA