RESONANCES AS VISCOSITY LIMITS FOR BLACK BOX PERTURBATIONS

HAOREN XIONG

Abstract. We show that the complex absorbing potential (CAP) method for computing scattering resonances applies to an abstractly defined class of black box perturbations of the Laplacian in $\mathbb{R}^n$ which can be analytically extended from $\mathbb{R}^n$ to a conic neighborhood in $\mathbb{C}^n$ near infinity. The black box setting allows a unifying treatment of diverse problems ranging from obstacle scattering to scattering on finite volume surfaces.

1. Introduction and statement of results

The complex absorbing potential (CAP) method has been used as a computational tool for finding scattering resonances – see Riss–Meyer [RiMe95] and Seideman–Miller [SeMi92] for an early treatment and Jagau et al [J*14] for some recent developments. Zworski [Zw18] showed that scattering resonances of $-\Delta+V$, $V\in L^\infty_{\rm comp}$, are limits of eigenvalues of $-\Delta+V-i\varepsilon x^2$ as $\varepsilon\to 0+$. The situation is very different for potentials of the Wigner–von Neumann type, in which case Kameoka and Nakamura [KaNa20] showed that the corresponding limits exist away from a discrete set of thresholds. Using an approach closer to [KaNa20] than [Zw18], the author extended Zworski's result to potentials which are exponentially decaying [Xi20]. In this paper we show that the CAP method is also valid for an abstractly defined class of *black box* perturbations of the Laplacian in $\mathbb{R}^n$ which can be analytically extended from $\mathbb{R}^n$ to a conic neighborhood in $\mathbb{C}^n$ near infinity.

We formulate black box scattering using the abstract setting introduced by Sjöstrand and Zworski in [SjZw91] except that the operator P is not assumed to be equal to $-\Delta$ near infinity. For that we follow Sjöstrand [Sj97] and assume that P is a dilation analytic perturbation of $-\Delta$ near infinity. The black box formalism allows an abstract treatment of diverse scattering problems without addressing the details of specific situations – see Examples 1–3 later in this section. We recall the setup as follows:

Let $\mathcal{H}$ be a complex separable Hilbert space with an orthogonal decomposition:

$$\mathcal{H}=\mathcal{H}_{R_0}\oplus L^2(\mathbb{R}^n\setminus B(0,R_0)),\tag{1.1}$$

where $B(x,R)=\{y\in\mathbb{R}^n:|x-y|<R\}$ and R_0 is fixed. The corresponding orthogonal projections will be denoted by $u\mapsto u|_{B(0,R_0)}$, and $u\mapsto u|_{\mathbb{R}^n\setminus B(0,R_0)}$ or simply by the

characteristic function 1_L of the corresponding set L. We consider an unbounded self-adjoint operator

$$P : \mathcal{H} \to \mathcal{H} \quad \text{with domain } \mathcal{D}. \tag{1.2}$$

We assume that

$$\mathcal{D}|_{\mathbb{R}^n \setminus B(0,R_0)} \subset H^2(\mathbb{R}^n \setminus B(0,R_0)), \tag{1.3}$$

and conversely, $u \in \mathcal{D}$ if $u \in H^2(\mathbb{R}^n \setminus B(0,R_0))$ and u vanishes near $B(0,R_0)$; and that

$$1_{B(0,R_0)}(P+i)^{-1} \text{ is compact.} \tag{1.4}$$

We also assume that,

$$\begin{gathered} 1_{\mathbb{R}^n \setminus B(0,R_0)} P u = Q(u|_{\mathbb{R}^n \setminus B(0,R_0)}), \qquad \text{for all } u \in \mathcal{D}, \\ Q = -\sum_{j,k=1}^n \partial_{x_j}(g^{jk}(x)\partial_{x_k}) + c(x), \quad g^{jk}, c \in \mathcal{C}_b^\infty(\mathbb{R}^n). \end{gathered} \tag{1.5}$$

Here $\mathcal{C}_b^\infty$ denotes the space of $\mathcal{C}^\infty$ functions with all derivatives bounded. Note that if $\psi \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ is constant near $B(0,R_0)$, then there is a natural way to define the multiplication: $\mathcal{H} \ni u \mapsto \psi u \in \mathcal{H}$, and we have $\psi u \in \mathcal{D}$ if $u \in \mathcal{D}$.

We make the further assumptions on the coefficients of Q: g^{jk}, c are real-valued functions on $\mathbb{R}^n$ satisfying

$$\begin{gathered} g^{jk} = g^{kj}, \forall\, j,k, \quad |\sum_{j,k=1}^n g^{jk}(x)\xi_j\xi_k| \geq C^{-1}|\xi|^2, \\ \sum_{j,k=1}^n g^{jk}(x)\xi_j\xi_k + c(x) \to \xi^2, \; |x| \to \infty. \end{gathered} \tag{1.6}$$

We will use the method of complex scaling – see §2.1 to define the resonances of P. For that we follow [Sj97] to make the following assumptions:

$$\begin{gathered} \text{There exist } \theta_0 \in [0, \pi/8], \; \delta > 0, \text{ and } R \geq R_0, \text{ such that} \\ \text{the coefficients } g^{jk}(x), c(x) \text{ of } Q \text{ extend analytically in } x \text{ to} \\ \{s\omega : \omega \in \mathbb{C}^n, \, \mathrm{dist}(\omega, \mathbb{S}^{n-1}) < \delta, \; s \in \mathbb{C}, \, |s| > R, \, \arg s \in (-\delta, \theta_0 + \delta)\} \\ \text{and the second half of (1.6) remains valid in this larger set.} \end{gathered} \tag{1.7}$$

We can now define the resonances z_j of P in $\mathbb{C} \setminus e^{-2i\theta_0}[0,\infty)$ as the eigenvalues of P on a suitable contour in $\mathbb{C}^n$, this set consists of the negative eigenvalues of P plus a discrete set in the sector $\{z \in \mathbb{C} \setminus \{0\} : -2\theta_0 < \arg z \leq 0\}$, see [SjZw91] and §2.1.

We now introduce a *regularized* operator,

$$P_\varepsilon := P - i\varepsilon(1-\chi(x))x^2, \quad \varepsilon > 0, \tag{1.8}$$

where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to 1 near $\overline{B(0,R_0)}$; $x^2 := x_1^2 + \cdots + x_n^2$. It follows from §3 that P_ε is an unbounded operator on $\mathcal{H}$ with a discrete spectrum. We have

Theorem 1. *Denote by* $\mathrm{Res}(P)$ *the set of resonances of* P. *Then, uniformly on any precompact open subset* Ω *of the sector* $\{z \in \mathbb{C} \setminus \{0\} : -2\theta_0 < \arg z < 3\pi/2 + 2\theta_0\}$,

$$\lim_{\varepsilon \to 0+} \mathrm{Spec}(P_\varepsilon) \cap \Omega = \mathrm{Res}(P) \cap \Omega,$$

where the limit is taken with respect to the Hausdorff metric, that is for two non-empty subsets A, B *of* $\mathbb{C}$,

$$d_H(A,B) := \max\left\{ \sup_{a \in A} \inf_{b \in B} |a-b|, \sup_{b \in B} \inf_{a \in A} |a-b| \right\}.$$

Remark: A more precise version of this theorem will be proved in §6, which involves the multiplicities of resonances z_j and eigenvalues $z_j(\varepsilon)$ defined in §2.1 and §3 respectively.

We refer to these limits as viscosity limits by analogy to the case of Pollicott–Ruelle resonances in Dyatlov–Zworski [DyZw15]. In that case, the analogue of P_ε is given by $X + \varepsilon\Delta$ where X (the analogue of our iP) is the generator of an Anosov flow on a compact manifold and Δ, the Laplace–Beltrami operator for some metric, is an analogue of our $|x|^2$ (on the Fourier transform side as in [KaNa20]). This then corresponds to a standard "viscosity/stochastic" regularization.

Fixed complex absorbing potentials have already been used in mathematical literature on scattering resonances. Stefanov [St05] showed that semiclassical resonances close to the real axis can be well approximated using eigenvalues of the Hamiltonian modified by a complex absorbing potential. For applications of fixed complex absorbing potentials in generalized geometric settings see for instance Nonnenmacher–Zworski [NoZw09], [NoZw15] and Vasy [Va13]. The analogous results to Theorem 1 were proved for Pollicott–Ruelle resonances in [DyZw15], for kinetic Brownian motion by Drouot [Dr17], for gradient flows by Dang–Rivière [DaRi17] (following earlier work of Frenkel–Losev–Nekrasov [FLN11]), and for 0th order pseudodifferential operators, motivated by problems in fluid mechanics, by Galkowski–Zworski [GaZw19].

Example 1. Obstacle scattering. Suppose that $\mathcal{O} \subset \overline{B(0,R_0)}$ is an open set such that $\partial\mathcal{O}$ is a smooth hypersurface in $\mathbb{R}^n$ and that $\mathbb{R}^n \setminus \mathcal{O}$ is connected. Let $\mathcal{H} = L^2(\mathbb{R}^n \setminus \mathcal{O})$, and $P = -\Delta|_{\mathbb{R}^n \setminus \mathcal{O}}$ on the exterior domain realized with any self-adjoint boundary conditions on $\partial\mathcal{O}$. For instance, the Dirichlet boundary condition

$$\mathcal{D} = \{u \in H^2(\mathbb{R}^n \setminus \mathcal{O}) : u|_{\partial\mathcal{O}} = 0\}$$

or the Neumann/Robin boundary condition

$$\mathcal{D} = \{u \in H^2(\mathbb{R}^n \setminus \mathcal{O}) : \partial_\nu u + \eta u|_{\partial\mathcal{O}} = 0\}$$

where ∂_ν is the normal derivative with respect to $\partial\mathcal{O}$ and η is a real-valued smooth function on $\partial\mathcal{O}$. Theorem 1 shows that the eigenvalues of $P - i\varepsilon x^2$ converge to the resonances of P (the irrelevance of the missing $i\varepsilon\chi(x)x^2$ term comes from continuity of resonances under compactly supported perturbations – see Stefanov [St94]).

Example 2. Scattering on asymptotically Euclidean space. Let M be a real analytic manifold which is diffeomorphic to $\mathbb{R}^n$ near infinity and equipped with a real analytic metric g which is asymptotically Euclidean. More precisely, let $g_{ij} = \delta_{ij} + h_{ij}$ be the metric tensor then we assume that $h_{ij}(x)$ extend analytically in x to

$$\{s\omega : \omega \in \mathbb{C}^n,\ \mathrm{dist}(\omega, \mathbb{S}^{n-1}) < \delta,\ s \in \mathbb{C},\ |s| > R,\ \arg s \in (-\delta, \theta_0 + \delta)\}$$

for some $\theta_0 \in [0, \pi/8]$, $\delta > 0$, $R \geq R_0$, and that $h_{ij} \to 0$ in this larger set. We put $P = -\Delta_g$, the Laplace–Beltrami operator with respect to the metric g, then all the black box assumptions are satisfied. Suppose that $\chi \in \mathcal{C}_c^\infty(M; [0,1])$ is equal to 1 near some compact set K and that $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus \overline{B(0, R_0)}$. Then the operator $-\Delta_g - i\varepsilon(1 - \chi(x))x^2$ has a discrete spectrum for $\varepsilon > 0$ and the eigenvalues converge to the resonances of $-\Delta_g$ uniformly on compact subsets of $-2\theta_0 < \arg z < 3\pi/2 + 2\theta_0$.

Example 3. Scattering on finite volume surfaces. This example was already discussed in [Zw18] but this paper provides a complete proof via the black box setting. Consider the modular surface $M = SL_2(\mathbb{Z})\backslash\mathbb{H}^2$ (or any surfaces with cusps – see [DyZw19, §4.1, Example 3]) equipped with the Poincaré metric g and $\Delta_M \leq 0$ the Laplacian on M. We choose the fundamental domain of $SL_2(\mathbb{Z})$ to be $\{x + iy \in \mathbb{H}^2 : |x| \leq 1/2, x^2 + y^2 \geq 1\}$ then Δ_M in the cusp $y > 1$ is given by $y^2(\partial_x^2 + \partial_y^2)$. Let $r = \log y$, $\theta = 2\pi x$, then M in (r, θ) coordinates admits the following decomposition:

$$M = M_0 \cup M_1,\ (M_1, g|_{M_1}) = ([0, \infty)_r \times \mathbb{S}^1_\theta,\ dr^2 + (2\pi)^{-2}e^{-2r}d\theta^2),\ \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

We recall the black box setup in this case from [DyZw19, §4.1, Example 3]. Let

$$\mathcal{H} = \mathcal{H}_0 \oplus L^2([0, \infty), dr), \quad \mathcal{H}_0 = L^2(M_0) \oplus \mathcal{H}_0^0,$$

where (with $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$)

$$\mathcal{H}_0^0 = \left\{ \{a_n(r)\}_{n \in \mathbb{Z}^*} : a_n \in L^2([0, \infty)),\ \sum_{n \in \mathbb{Z}^*} \int_0^\infty |a_n(r)|^2 dr < \infty \right\}.$$

We can identify $L^2(M)$ with $\mathcal{H}$ via the following isomorphism:

$$\iota : L^2(M) \ni u \mapsto \big(u|_{M_0}, \{e^{-r/2}u_n(r)\}_{n \in \mathbb{Z}^*}, e^{-r/2}u_0(r)\big) \in \mathcal{H},$$
$$u_n(r) := \frac{1}{2\pi} \int_{\mathbb{S}^1} u(r, \theta)e^{-in\theta}d\theta, \quad r > 0.$$

Then $P := -\Delta_M - 1/4$ is a black box Hamiltonian on $\mathcal{H}$ which equals $-\partial_r^2$ on $L^2([0, \infty), dr)$ – see [DyZw19, §4.1, Example 3]. In the language of Theorem 1 and in

(x, y) coordinates

$$P_\varepsilon = -\Delta_M - 1/4 - i\varepsilon(1-\chi(y))(\log y)^2\Pi_0, \quad \Pi_0 u(x,y) := \int_{-1/2}^{1/2} u(x', y)\, dx'.$$

where $\chi \in \mathcal{C}_c^\infty([0,\infty))$, $\chi(y) \equiv 1$ for $y < 2$ and $\chi(y) \equiv 0$ for $y > 3$. The eigenvalues of P_ε converge to the resonances of P uniformly on compact subsets of $\arg z > -\pi/4$. Equivalently if we define $s(\varepsilon) \in \Sigma_\varepsilon \Leftrightarrow s(\varepsilon)(1-s(\varepsilon)) - 1/4 \in \operatorname{Spec}(P_\varepsilon)$, then the limit points of Σ_ε, $\varepsilon \to 0+$, in $\operatorname{Re} s < 1/2$, $\arg(s - 1/2) \neq 11\pi/8$ are given by the nontrivial zeros of $\zeta(2s)$ where ζ is the Riemann zeta function – see [Zw18, Example 2] and [DyZw19, §4.4 Example 3].

The paper is organized as follows. In §2.1 we review the method of complex scaling and define the resonances of P as the eigenvalues of the complex scaled operator $\mathcal{P}_\theta$. In §3 we show that P_ε has a discrete spectrum in $\mathbb{C} \setminus e^{-i\pi/4}[0,\infty)$, which is invariant under complex scaling. Since our operator is an abstract perturbation of $-\Delta$, in §4 we use a different method from [Zw18] and [Xi20] to characterize the eigenvalues of $\mathcal{P}_{\varepsilon,\theta}$, $\varepsilon \geq 0$. More precisely, we use a reference operator reviewed in §2.2 to introduce the Dirichlet-to-Neumann operator $\mathcal{N}_{\varepsilon,\theta}(z)$ associated with $\mathcal{P}_{\varepsilon,\theta}$ and an artificial smooth obstacle $\mathcal{O}$. The artificial obstacle problem is needed to separate the abstract black box from the differential operator outside. The operator $\mathcal{N}_{\varepsilon,\theta}(z)$ is well-defined for all z except for a discrete set depending on the obstacle, and we show that the eigenvalues of $\mathcal{P}_{\varepsilon,\theta}$ can be identified with the poles of $z \mapsto \mathcal{N}_{\varepsilon,\theta}(z)^{-1}$, with agreement of multiplicities. In §5 we show that the obstacle can be chosen so that the corresponding $\mathcal{N}_{\varepsilon,\theta}(z)$ is well-defined near the resonances z_j. The proof of Theorem 1 is completed in §6 by obtaining further estimates on $\mathcal{N}_{\varepsilon,\theta}(z)$.

Notation. We use the following notation: $f = O_\ell(g)_H$ means that $\|f\|_H \leq C_\ell g$ where the norm (or any seminorm) is in the space H, and the constant C_ℓ depends on ℓ. When either ℓ or H are absent then the constant is universal or the estimate is scalar, respectively. When $G = O_\ell(g) : H_1 \to H_2$ then the operator $G : H_1 \to H_2$ has its norm bounded by $C_\ell g$.

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2. Preliminaries

2.1. **Review of Complex Scaling.** Complex scaling has been a standard technique in resonance theory since the works of Aguilar–Combes [AgCo71], Balslev–Combes

[BaCo71] and Simon [Si79]. Here we follow rather closely the presentation in [Sj97] since our assumptions on the operator P is weaker than [SjZw91].

A smooth submanifold $\Gamma \subset \mathbb{C}^n$ is said to be totally real if $T_x\Gamma \cap iT_x\Gamma = \{0\}$ for every $x \in \Gamma$, where we identify $T_x\Gamma$ with a real subspace of $T_x\mathbb{C}^n \simeq \mathbb{C}^n$. We say that Γ is maximally totally real if Γ is totally real and of maximal (real) dimension n, the natural example is $\Gamma = \mathbb{R}^n$. Let $\Gamma \subset \mathbb{C}^n$ be smooth and of real dimension n, then locally Γ can be represented using real coordinates: $\mathbb{R}^n \ni x \mapsto f(x) \in \Gamma$. Let $\tilde{f}$ be an almost analytic extension of f so that $\bar{\partial}\tilde{f}$ vanishes to infinite order on $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$, then since $d\tilde{f}(x)$ is complex linear, $iT_{f(x)}\Gamma = d\tilde{f}(x)(iT_x\mathbb{R}^n)$. Hence Γ is totally real in a neighborhood of $f(x)$ if and only if $d\tilde{f}(x)$ is injective, i.e. $\det df(x) \neq 0$.

Let $\Omega \subset \mathbb{C}^n$ be an open neighborhood of Γ such that Γ is closed in Ω, and let

$$A(z, D_z) = \sum_{|\alpha| \le m} a_\alpha(z) D_z^\alpha, \quad D_{z_j} := \frac{1}{i}\partial_{z_j}, \quad D_z^\alpha = D_{z_1}^{\alpha_1} \cdots D_{z_n}^{\alpha_n},$$

be a differential operator on Ω with holomorphic coefficients. Define $A_\Gamma : \mathcal{C}^\infty(\Gamma) \to \mathcal{C}^\infty(\Gamma)$ by

$$A_\Gamma u = (A\tilde{u})|_\Gamma, \tag{2.1}$$

where $\tilde{u}$ is an almost analytic extension of u, that is, a smooth extension of u to a neighborhood of Γ such that $\bar{\partial}\tilde{u}$ vanishes to infinite order on Γ. A_Γ is then a differential operator on Γ with smooth coefficients, and for the principal symbols we have

$$a_\Gamma = a|_{T^*\Gamma},$$

where a is the principal symbol of A.

We recall a deformation result from [SjZw91, Lemma 3.1]:

Lemma 2.1. *Suppose that $W \subset \mathbb{R}^n$ is open and that $F : [0,1] \times W \ni (s,x) \mapsto F(s,x) \in \mathbb{C}^n$, is a smooth proper map satisfying for all $s \in [0,1]$*

$$\det \partial_x F(s,x) \neq 0, \quad \textit{and } x \mapsto F(s,x) \textit{ is injective,}$$

and assume that $x \in W \setminus K \implies F(s,x) = F(0,x)$ for some compact $K \subset W$.

Let $A(z, D_z)$ be a differential operator with holomorphic coefficients defined in a neighborhood of $F([0,1] \times W)$ such that for $0 \le s \le 1$ and $\Gamma_s := F(\{s\} \times W)$, A_{Γ_s} is elliptic.

If $u_0 \in \mathcal{C}^\infty(\Gamma_0)$ and $A_{\Gamma_0}u_0$ extends to a holomorphic function in a neighborhood of $F([0,1] \times W)$, then the same holds for u_0.

The lemma will be applied to a family of deformations of $\mathbb{R}^n$ in $\mathbb{C}^n$. We aim to restrict the operators P_ε, $\varepsilon \ge 0$, to the corresponding totally real submanifolds. For given $\alpha_0 > 0$ and $R_1 > R_0$, we can construct a smooth function

$$[0, \theta_0] \times [0, \infty) \ni (\theta, t) \mapsto g_\theta(t) \in \mathbb{C},$$

injective for every θ, with the following properties:

(i) $g_\theta(t) = t$ for $0 \le t \le R_1$,
(ii) $0 \le \arg g_\theta(t) \le \theta, \quad \partial_t g_\theta(t) \neq 0$,
(iii) $\arg g_\theta(t) \le \arg \partial_t g_\theta(t) \le \arg g_\theta(t) + \alpha_0$,
(iv) $g_\theta(t) = e^{i\theta} t$ for $t \ge T_0$, where T_0 depends only on α_0 and R_1.

We now define the totally real submanifolds, Γ_θ, as images of $\mathbb{R}^n$ under the maps

$$f_\theta : \mathbb{R}^n \ni x = t\omega \mapsto g_\theta(t)\omega \in \mathbb{C}^n, \ t = |x|.$$

Then a dilated operator $\mathcal{P}_\theta$ can be defined as follows. Let

$$\mathcal{H}_\theta = \mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0, R_0)),$$

where $B(0, R_0)$ denotes the real ball as before. If $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(B(0, R_1))$ is equal to 1 near $\overline{B(0, R_0)}$, we put

$$\mathcal{D}_\theta = \{u \in \mathcal{H}_\theta : \chi u \in \mathcal{D}, \ (1-\chi)u \in H^2(\Gamma_\theta \setminus B(0, R_0))\}.$$

Let $\mathcal{P}_\theta$ be the unbounded operator $\mathcal{H}_\theta \to \mathcal{H}_\theta$ with domain $\mathcal{D}_\theta$, given by

$$\mathcal{P}_\theta u := P(\chi u) + Q_\theta((1-\chi)u), \quad Q_\theta := -\sum_{j,k=1}^{n} (\partial_{z_j}(g^{jk}(z)\partial_{z_k}) + c(z))|_{\Gamma_\theta}.$$

These definitions do not depend on the choice of χ.

We recall some properties of the dilated Laplacian from [SjZw91, §3]. Let

$$\Delta_\theta := (\Delta_z)|_{\Gamma_\theta}, \quad x_\theta := z|_{\Gamma_\theta}.$$

Parametrizing Γ_θ by $[0, \infty) \times \mathbb{S}^{n-1} \ni (t, \omega) \mapsto g_\theta(t)\omega$, we obtain

$$-\Delta_\theta = (g'_\theta(t)^{-1} D_t)^2 - i(n-1)(g_\theta(t) g'_\theta(t))^{-1} D_t + g_\theta(t)^{-2} D_\omega^2, \tag{2.2}$$

where $D_t = -i\partial_t$ and $D_\omega^2 = -\Delta_{\mathbb{S}^{n-1}}$. If ω^{*2} denotes the principal symbol of D_ω^2 and we let τ be the dual variable of t, then the principal symbol of $-\Delta_\theta$ is

$$\sigma(-\Delta_\theta) = g'_\theta(t)^{-2}\tau^2 + g_\theta(t)^{-2}\omega^{*2},$$

so pointwise on Γ_θ, $-\Delta_\theta$ is elliptic and the principal symbol takes values in an angle of size $\le 2\alpha_0$, while globally, $\sigma(-\Delta_\theta)$ takes values in the sector $-2\theta - 2\alpha_0 \le \arg z \le 0$. The basic result based on ellipticity at infinity is

$$\begin{gathered} -2\theta + \delta < \arg z < 2\pi - 2\theta - \delta, \quad |z| > \delta \implies \\ (-\Delta_\theta - z)^{-1} = O_\delta(|z|^{\frac{j-2}{2}}) : L^2(\Gamma_\theta) \to H^j(\Gamma_\theta), \ j = 0, 1, 2. \end{gathered} \tag{2.3}$$

This follows from [SjZw91, Lemmas 3.2–3.5 and §4] applied with $P = -\Delta$.

$\mathcal{P}_\theta$, as a perturbation of $-\Delta_\theta$, is also elliptic – see [Sj97, §5]. More precisely, choosing R_1 large enough, it follows from the assumptions (1.6) and (1.7) that

$$\begin{gathered}\text{In } \Gamma_\theta \setminus B(0,R_0),\ \mathcal{P}_\theta \text{ is an elliptic differential operator whose principal}\\ \text{symbol pointwise on } \Gamma_\theta \text{ takes its values in an angle of size} \leq 3\alpha_0,\\ \text{and globally in a sector } -2\theta - 3\alpha_0 \leq \arg z \leq \alpha_0.\end{gathered} \tag{2.4}$$

$$\begin{gathered}\text{The coefficients of } \mathcal{P}_\theta - e^{-2i\theta}(-\Delta) \text{ tend to zero when } \Gamma_\theta \ni x \to \infty,\\ \text{where we identify } \Gamma_\theta \text{ and } \mathbb{R}^n, \text{ by means of } f_\theta.\end{gathered} \tag{2.5}$$

We recall some basic results about $\mathcal{P}_\theta$ from [Sj97, §5]:

Lemma 2.2. *If $z \in \mathbb{C} \setminus \{0\}$, $\arg z \neq -2\theta$, then $\mathcal{P}_\theta - z : \mathcal{D}_\theta \to \mathcal{H}_\theta$ is a Fredholm operator of index 0. In particular the spectrum of $\mathcal{P}_\theta$ in $\mathbb{C} \setminus e^{-2i\theta}[0,\infty)$ is discrete.*

Proof. The first part of the lemma is the same as Lemma 7.3 in the lecture notes by Sjöstrand [Sj02], the corresponding proof can be found there. It remains to show that $\mathcal{P}_\theta$ has a discrete spectrum in $\mathbb{C} \setminus e^{-2i\theta}[0,\infty)$. For that, let $z_0 = iL,\ L \geq 1$, we put

$$E(z_0) = \tilde{\chi}_1(P - z_0)^{-1}\chi_1 + (1-\chi_0)(-\Delta_\theta - z_0)^{-1}(1-\chi_1), \tag{2.6}$$

where $\chi_1 \in \mathcal{C}_c^\infty(B(0,R_1))$ is equal to 1 near $\operatorname{supp}\chi_0$ and $\chi_0 = 1$ on $B(0,R_1-\delta)$, for some $\delta > 0$ small. Then we have

$$(\mathcal{P}_\theta - z_0)E(z_0) = I + K(z_0) + K_1(z_0),$$

where

$$\begin{gathered}K(z_0) = [P,\tilde{\chi}_1](P-z_0)^{-1}\chi_1 + [\Delta_\theta,\chi_0](-\Delta_\theta - z_0)^{-1}(1-\chi_1),\\ K_1(z_0) = (\mathcal{P}_\theta - (-\Delta_\theta))(1-\chi_0)(-\Delta_\theta - z_0)^{-1}(1-\chi_1).\end{gathered}$$

Choosing R_1 sufficiently large, we may assume by (2.3) and (2.5) that $\|K_1(z_0)\|_{\mathcal{H}_\theta \to \mathcal{H}_\theta} \leq 1/2$, for all $z_0 = iL,\ L \geq 1$. Then we get

$$(\mathcal{P}_\theta - z_0)E(z_0)(I + K_1(z_0))^{-1} = I + K(z_0)(I + K_1(z_0))^{-1}.$$

It follows from (2.3) that $K(iL) = O(L^{-1/2}) : \mathcal{H}_\theta \to \mathcal{H}_\theta$, then for $z_0 = iL,\ L \gg 1$, $\|K(z_0)(I+K_1(z_0))^{-1}\|_{\mathcal{H}_\theta \to \mathcal{H}_\theta} \leq 1/2$, thus $\mathcal{P}_\theta - z_0$ has a right inverse:

$$E(z_0)(I + K_1(z_0))^{-1}(I + K(z_0)(I + K_1(z_0))^{-1})^{-1},$$

which implies that $\mathcal{P}_\theta - z_0$ is surjective. Since $\mathcal{P}_\theta - z_0$ is a Fredholm operator of index 0, it must also be injective. Hence by the inverse mapping theorem, $\mathcal{P}_\theta - z_0$ is invertible and we have

$$(\mathcal{P}_\theta - z_0)^{-1} = E(z_0)(I + K_1(z_0))^{-1}(I + K(z_0)(I + K_1(z_0))^{-1})^{-1}. \tag{2.7}$$

Analytic Fredholm theory then shows that $\mathcal{P}_\theta$ has a discrete spectrum. ∎

Lemma 2.3. *Assume that* $0 \le \theta_1 < \theta_2 \le \theta_0$ *and let* $z_0 \in \mathbb{C} \setminus e^{-2i[\theta_1,\theta_2]}[0,\infty)$. *Then*

$$\dim\ker(\mathcal{P}_{\theta_1} - z_0) = \dim\ker(\mathcal{P}_{\theta_2} - z_0).$$

This is identical to [SjZw91, Lemma 3.4] and the proof is the same as there using Lemma 2.1.

Lemma 2.3 shows that the spectrum in the sector $-2\theta_0 < \arg z \le 0$ is independent of θ in the following sense: We say that $z \in \mathbb{C} \setminus \{0\}$, $-2\theta_0 < \arg z \le 0$ is a resonance for P if and only if $z \in \operatorname{Spec}(\mathcal{P}_\theta)$ with $-2\theta < \arg z \le 0$ for some $\theta \in (0, \theta_0]$. For such a resonance $z_0 \in e^{-2i[0,\theta)}(0,\infty)$, the spectral projection

$$\Pi_\theta(z_0) = \frac{1}{2\pi i} \oint_{z_0} (z - \mathcal{P}_\theta)^{-1} dz, \tag{2.8}$$

where the integral is over a positively oriented circle enclosing z_0 and containing no resonances other than z_0, is of finite rank. The restriction of $\mathcal{P}_\theta - z_0$ to $\operatorname{Ran}\Pi_\theta(z_0)$ is nilpotent. If $\tilde\theta \in [0, \theta_0]$ is a second number with $z_0 \in e^{-2i[0,\tilde\theta)}(0,\infty)$, then since Lemma 2.3 can be extended to $\dim\ker(\mathcal{P}_\theta - z_0)^k = \dim\ker(\mathcal{P}_{\tilde\theta} - z_0)^k$ for all k, $\Pi_\theta(z_0)$ and $\Pi_{\tilde\theta}(z_0)$ have the same rank, which by definition is the multiplicity of the resonance z_0:

$$m(z_0) := \operatorname{rank}\Pi_\theta(z_0), \quad -2\theta < \arg z_0 \le 0. \tag{2.9}$$

2.2. **A reference operator.** As explained in §1, to separate the abstract black box from the differential operator outside we introduce a *reference operator* $P^{\mathcal{O}}$ associated with a bounded open set $\mathcal{O} \subset \mathbb{R}^n$ containing $\overline{B(0,R_0)}$. We assume that $\partial\mathcal{O}$ is a smooth hypersurface in $\mathbb{R}^n$. In the notation of (1.1), we put

$$\mathcal{H}^{\mathcal{O}} := \mathcal{H}_{R_0} \oplus L^2(\mathcal{O} \setminus B(0,R_0)). \tag{2.10}$$

The corresponding orthogonal projections are denoted by

$$u \mapsto 1_{B(0,R_0)}\, u = u|_{B(0,R_0)}, \quad u \mapsto 1_{\mathcal{O}\setminus B(0,R_0)}\, u = u|_{\mathcal{O}\setminus B(0,R_0)}.$$

If P is a black box Hamiltonian introduced in §1 with domain $\mathcal{D}$, then we define

$$\begin{aligned} \mathcal{D}^{\mathcal{O}} := \{u \in \mathcal{H}^{\mathcal{O}} : \psi \in \mathcal{C}^\infty_{\rm c}(\mathcal{O}),\ \psi = 1 \text{ near } \overline{B(0,R_0)} \Rightarrow \\ \psi u \in \mathcal{D},\ (1-\psi)u \in H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})\} \end{aligned} \tag{2.11}$$

and, for any ψ with the property (2.11),

$$\begin{aligned} &P^{\mathcal{O}} : \mathcal{D}^{\mathcal{O}} \to \mathcal{H}^{\mathcal{O}}, \\ &P^{\mathcal{O}} u := P(\psi u) + Q((1-\psi)u). \end{aligned} \tag{2.12}$$

Assumptions (1.3), (1.5) show that this definition is independent of the choice of ψ.

We recall some basic properties of the reference operator from [SjZw91, §7]:

Lemma 2.4. *Suppose that $\mathcal{O} \subset \mathbb{R}^n$ is an open set containing $\overline{B(0,R_0)}$ such that $\partial\mathcal{O}$ is a smooth hypersurface in $\mathbb{R}^n$. Let $P^{\mathcal{O}}$ be the reference operator defined in (2.12). Then, with $\mathcal{H}^{\mathcal{O}}$ given by (2.10),*

$$P^{\mathcal{O}} : \mathcal{H}^{\mathcal{O}} \to \mathcal{H}^{\mathcal{O}},$$

is a self-adjoint operator with domain $\mathcal{D}^{\mathcal{O}}$ defined in (2.11). Furthermore, the resolvent $(P^{\mathcal{O}}+i)^{-1}$ is compact and thus $P^{\mathcal{O}}$ has discrete spectrum which is contained in $\mathbb{R}$.

For the proof we refer to Dyatlov–Zworski [DyZw19, Lemma 4.11] and we remark that the arguments there is still valid if we replace the assumption there: $P = -\Delta$ in $\mathbb{R}^n \setminus B(0,R_0)$, by the assumption (1.5).

3. The regularized operator

In this section we show that the spectrum of P_ε is invariant under complex scaling. Choosing R_1 such that $\operatorname{supp}\chi \subset B(0,R_1)$ when we construct the complex contours Γ_θ, the complex absorbing potential $-i\varepsilon(1-\chi(x))x^2$ can be analytically extended to Γ_θ, thus it defines a multiplication on the following subspace of $\mathcal{H}_\theta$:

$$\widehat{\mathcal{H}}_\theta := \mathcal{H}_{R_0} \oplus |x_\theta|^{-2} L^2(\Gamma_\theta \setminus B(0,R_0)),$$

where $x_\theta := f_\theta(x)$ denotes the parametrization of Γ_θ. We now introduce the deformation of P_ε on Γ_θ, $\theta \in [0,\theta_0)$:

$$\mathcal{P}_{\varepsilon,\theta} := \mathcal{P}_\theta - i\varepsilon(1-\chi(x_\theta))x_\theta^2, \quad \text{with the domain } \widehat{\mathcal{D}}_\theta := \mathcal{D}_\theta \cap \widehat{\mathcal{H}}_\theta. \tag{3.1}$$

It follows from (2.5) that $\mathcal{P}_{\varepsilon,\theta}$ near infinity is close to the operator

$$H_{\varepsilon,\theta} := -e^{-2i\theta}\Delta - i\varepsilon e^{2i\theta}x^2, \tag{3.2}$$

which was considered by Davies [Da99] as an interesting example of a non-normal differential operator. We recall the following basic result:

Lemma 3.1. *For $\varepsilon > 0$, $0 \le \theta \le \pi/8$, $H_{\varepsilon,\theta}$ is a closed densely defined operator on $L^2(\mathbb{R}^n)$ equipped with the domain $H^2(\mathbb{R}^n) \cap \langle x\rangle^{-2} L^2(\mathbb{R}^n)$. The spectrum is given by*

$$\operatorname{Spec}(H_{\varepsilon,\theta}) = \{e^{-i\pi/4}\sqrt{\varepsilon}(2|\alpha|+n) : \alpha \in \mathbb{N}_0^n\}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_n. \tag{3.3}$$

In addition for any $\delta > 0$ we have uniformly in $\varepsilon > 0$,

$$\begin{gathered}(H_{\varepsilon,\theta} - z)^{-1} = O_\delta(|z|^{\frac{j-2}{2}}) : L^2(\mathbb{R}^n) \to H^j(\mathbb{R}^n), \quad j = 0,1,2,\\ \text{for } -2\theta + \delta < \arg z < 3\pi/2 + 2\theta - \delta, \quad |z| > \delta.\end{gathered} \tag{3.4}$$

Proof. For every $\varepsilon > 0$ and $0 \le \theta \le \pi/8$, $H_{\varepsilon,\theta}$ can be viewed as the Weyl quantization of the complex-valued quadratic form $q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C}$, $\quad (x,\xi) \mapsto e^{-2i\theta}\xi^2 - i\varepsilon e^{2i\theta}x^2$, which shall be viewed as a closed densely defined operator on $L^2(\mathbb{R}^n)$ equipped with the domain $\mathcal{D}(H_{\varepsilon,\theta}) := \{u \in L^2(\mathbb{R}^n) : H_{\varepsilon,\theta}u \in L^2(\mathbb{R}^n)\}$. For the analysis of the spectrum for general quadratic operators see Hitrik–Sjöstrand–Viola [HSV13] and references given there, in particular we obtain (3.3). Noticing that the numerical range of q is the sector $\{z \in \mathbb{C} : 3\pi/2 + 2\theta \leq \arg z \leq 2\pi - 2\theta\}$, $H_{\varepsilon,\theta} - i$ is elliptic with respect to the order function $m = 1 + x^2 + \xi^2$ in the sense that $|q - i| \geq Cm$ for some $C = C(\varepsilon) > 0$. Since $H_{\varepsilon,\theta} - i$ is invertible by (3.3), we conclude that

$$(H_{\varepsilon,\theta} - i)^{-1} : L^2(\mathbb{R}^n) \to m^{-1}(x, D_x)L^2(\mathbb{R}^n) = H^2(\mathbb{R}^n) \cap \langle x\rangle^{-2}L^2(\mathbb{R}^n).$$

Hence $u \in \mathcal{D}(H_{\varepsilon,\theta}) \Rightarrow u = (H_{\varepsilon,\theta} - i)^{-1}(H_{\varepsilon,\theta}u - iu) \in H^2(\mathbb{R}^n) \cap \langle x\rangle^{-2}L^2(\mathbb{R}^n)$. Now we rescale $y = \sqrt{\varepsilon}x$, then $H_{\varepsilon,\theta}$ is unitary equivalent to $-e^{-2i\theta}\varepsilon\Delta_y - ie^{2i\theta}y^2$, that is a semiclassical quadratic operator with $h = \sqrt{\varepsilon}$. The bounds (3.4) follow from semiclassical ellipticity of $-e^{-2i\theta}\varepsilon\Delta_y - ie^{2i\theta}y^2 - z$ for $-2\theta + \delta < \arg z < 3\pi/2 + 2\theta - \delta$, $|z| > \delta$. $\square$

Then we show that $\mathcal{P}_{\varepsilon,\theta}$ is a Fredholm operator for $z \notin e^{-i\pi/4}[0, \infty)$.

Lemma 3.2. *If $z \in \mathbb{C} \setminus \{0\}$, $\arg z \neq -\pi/4$, then for each $\varepsilon > 0$ and $0 \leq \theta < \theta_0$, $\mathcal{P}_{\varepsilon,\theta} - z : \widehat{\mathcal{D}}_\theta \to \mathcal{H}_\theta$ is a Fredholm operator of index 0. In particular the spectrum of $\mathcal{P}_{\varepsilon,\theta}$ in $\mathbb{C} \setminus e^{-i\pi/4}[0, \infty)$ is discrete.*

Proof. We choose $\chi_j \in \mathcal{C}^\infty_c(\Gamma_\theta)$, $j = 0, 1, 2, 3$, such that $\chi_j = 1$ near $\operatorname{supp} \chi_{j-1}$ and that $\chi_0(g_\theta(t)\omega) = 1$ for any $t \leq T_0$, thus $1 - \chi_j$ are supported in the region where $\Gamma_\theta \ni x_\theta = e^{i\theta}x$, $x \in \mathbb{R}^n$. Lemma 3.1 then shows that if $\arg z \neq -\pi/4$,

$$(1 - \chi_0)(H_{\varepsilon,\theta} - z)^{-1}(1 - \chi_1) : \mathcal{H}_\theta \to \widehat{\mathcal{D}}_\theta.$$

Now we fix $z \in \mathbb{C} \setminus \{0\}$ with $\arg z \neq -\pi/4$. Using (2.5) we may assume that $\operatorname{supp} \chi_0$ is large enough so that $\|(\mathcal{P}_{\varepsilon,\theta} - H_{\varepsilon,\theta})(1 - \chi_0)(H_{\varepsilon,\theta} - z)^{-1}(1 - \chi_1)\|_{\mathcal{H}_\theta \to \mathcal{H}_\theta} \leq 1/2$. Then we choose $z_0 = iL$, $L \gg 1$ using (2.7) such that $\|\varepsilon(\chi_3 - \chi)x_\theta^2(\mathcal{P}_\theta - z_0)^{-1}\|_{\mathcal{H}_\theta \to \mathcal{H}_\theta} \leq 1/2$, thus

$$(\mathcal{P}_\theta - i\varepsilon(\chi_3 - \chi)x_\theta^2 - z_0)^{-1} = (\mathcal{P}_\theta - z_0)^{-1}(I - i\varepsilon(\chi_3 - \chi)x_\theta^2(\mathcal{P}_\theta - z_0)^{-1})^{-1} \tag{3.5}$$

exists. We put

$$E(z) = \chi_2(\mathcal{P}_\theta - i\varepsilon(\chi_3 - \chi)x_\theta^2 - z_0)^{-1}\chi_1 + (1 - \chi_0)(H_{\varepsilon,\theta} - z)^{-1}(1 - \chi_1).$$

Then we get

$$(\mathcal{P}_{\varepsilon,\theta} - z)E(z) = I + K(z) + K_1(z),$$

where

$$\begin{gathered} K(z) = ((z_0 - z)\chi_2 + [\mathcal{P}_\theta, \chi_2])(\mathcal{P}_\theta - i\varepsilon(\chi_3 - \chi)x_\theta^2 - z_0)^{-1}\chi_1 \\ + [e^{-2i\theta}\Delta, \chi_0](H_{\varepsilon,\theta} - z)^{-1}(1 - \chi_1) \\ K_1(z) = (\mathcal{P}_{\varepsilon,\theta} - H_{\varepsilon,\theta})(1 - \chi_0)(H_{\varepsilon,\theta} - z)^{-1}(1 - \chi_1). \end{gathered}$$

Recalling that $\|K_1(z)\|_{\mathcal{H}_\theta \to \mathcal{H}_\theta} \leq 1/2$, we obtain that $I + K_1(z)$ is invertible thus

$$(\mathcal{P}_{\varepsilon,\theta} - z)E(z)(I + K_1(z))^{-1} = I + K(z)(I + K_1(z))^{-1}.$$

Since $(\mathcal{P}_\theta - z_0)^{-1} : \mathcal{H}_\theta \to \mathcal{D}_\theta$, we conclude that $K(z)$ is compact: $\mathcal{H}_\theta \to \mathcal{H}_\theta$. Hence $E(z)(I + K_1(z))$ is an approximate right inverse of $\mathcal{P}_{\varepsilon,\theta} - z$.

As an approximate left inverse, we put

$$F(z) = \chi_1(\mathcal{P}_\theta - i\varepsilon(\chi_3 - \chi)x_\theta^2 - z_0)^{-1}\chi_2 + (1 - \chi_1)(H_{\varepsilon,\theta} - z)^{-1}(1 - \chi_0).$$

Then

$$F(z)(\mathcal{P}_{\varepsilon,\theta} - z) = I + L(z) + L_1(z),$$

where

$$\begin{aligned} L(z) = \chi_1(\mathcal{P}_\theta - i\varepsilon(\chi_3 - \chi)x_\theta^2 - z_0)^{-1}((z_0 - z)\chi_2 - [\mathcal{P}_\theta, \chi_2]) \\ -(1 - \chi_1)(H_{\varepsilon,\theta} - z)^{-1}[e^{-2i\theta}\Delta, \chi_0] \\ L_1(z) = (1 - \chi_1)(H_{\varepsilon,\theta} - z)^{-1}(1 - \chi_0)(\mathcal{P}_{\varepsilon,\theta} - H_{\varepsilon,\theta}). \end{aligned}$$

We may assume again by (2.5) that $\|L_1(z)\|_{\widehat{\mathcal{D}}_\theta \to \widehat{\mathcal{D}}_\theta} \le 1/2$, then

$$(I + L_1(z))^{-1}F(z)(\mathcal{P}_{\varepsilon,\theta} - z) = I + (I + L_1(z))^{-1}L(z).$$

Using (1.3), we see that $[e^{-2i\theta}\Delta, \chi_0]$ is compact: $\widehat{\mathcal{D}}_\theta \to \mathcal{H}_\theta$, thus $L(z)$ is compact: $\widehat{\mathcal{D}}_\theta \to \widehat{\mathcal{D}}_\theta$, $(I + L_1(z))^{-1}F(z)$ is an approximate left inverse.

We have shown that $\mathcal{P}_{\varepsilon,\theta} - z : \widehat{\mathcal{D}}_\theta \to \mathcal{H}_\theta$ is a Fredholm operator. This operator depends continuously on (θ, z), thus the index is constant under deformation in (θ, z). Deforming z into i and θ down to 0, we see that the index of $\mathcal{P}_{\varepsilon,\theta} - z$ is equal to the index of $P_\varepsilon - i : \widehat{\mathcal{D}} \to \mathcal{H}$ (where we omit the subscript 0). Repeating the arguments above, we can also show that for every $\gamma \in [0, \pi/2]$, $P + e^{-i\gamma}\varepsilon(1 - \chi(x))x^2 - i : \widehat{\mathcal{D}} \to \mathcal{H}$ is a Fredholm operator. Deforming γ from $\pi/2$ (that is for P_ε) to 0, it follows that the index of $P_\varepsilon - i$ is equal to the index of $P + \varepsilon(1 - \chi(x))x^2 - i$, which is 0 since $P + \varepsilon(1 - \chi(x))x^2 : \widehat{\mathcal{D}} \to \mathcal{H}$ is self-adjoint, see [HSV13, §1]. Hence we conclude that $\mathcal{P}_{\varepsilon,\theta} - z$ is of index 0.

It remains to show that $\mathcal{P}_{\varepsilon,\theta}$ has a discrete spectrum in $\mathbb{C} \setminus e^{-i\pi/4}[0, \infty)$. Recalling first (3.5) and then (2.6), (2.7), we see that $\|K(z_0)\|_{\mathcal{H}_\theta \to \mathcal{H}_\theta}$ can be controlled by $\|(1 - \chi_0)(P - z_0)^{-1}\|_{\mathcal{H} \to H^1(\mathbb{R}^n)}$, $\|(-\Delta_\theta - z_0)^{-1}\|_{L^2 \to H^1}$ and $\|(H_{\varepsilon,\theta} - z_0)^{-1}\|_{L^2 \to H^1}$. It then follows from (2.3) and (3.4) that $K(iL) = O(L^{-1/2}) : \mathcal{H}_\theta \to \mathcal{H}_\theta$. Hence for $z_0 = iL$, $L \gg 1$, $I + K(z_0)(I + K_1(z_0))^{-1}$ is invertible and we have

$$(\mathcal{P}_{\varepsilon,\theta} - z_0)E(z_0)(I + K_1(z_0))^{-1}(I + K(z_0)(I + K_1(z_0))^{-1})^{-1} = I,$$

which implies that $\mathcal{P}_{\varepsilon,\theta} - z_0$ is surjective. Since $\mathcal{P}_{\varepsilon,\theta} - z_0$ is a Fredholm operator of index 0, it must also be injective. Thus $\mathcal{P}_{\varepsilon,\theta} - z_0$ is invertible by the inverse mapping theorem. Analytic Fredholm theory then shows that $\mathcal{P}_{\varepsilon,\theta}$ has a discrete spectrum. $\square$

Lemma 3.3. *For each $0 \le \theta < \theta_0$ and $\varepsilon > 0$, let $\psi \in \mathcal{C}_c^\infty(B(0, R_1); [0, 1])$ be equal to 1 near $\overline{B(0, R_0)}$ so that ψ is a function on Γ_θ and defines a multiplication on $\mathcal{H}_\theta$. Then* *we have, meromorphically in the region* $-\pi/4 < \arg z < 7\pi/4$,

$$\psi(P_\varepsilon - z)^{-1}\psi = \psi(\mathcal{P}_{\varepsilon,\theta} - z)^{-1}\psi. \tag{3.6}$$

Proof. We modify the proof of [Zw18, Lemma 2]. It is sufficient to show that for $0 \leq \theta_1 < \theta_2 < \theta_0$, $|\theta_1 - \theta_2| \ll 1$,

$$\psi(\mathcal{P}_{\varepsilon,\theta_1} - z)^{-1}\psi = \psi(\mathcal{P}_{\varepsilon,\theta_2} - z)^{-1}\psi. \tag{3.7}$$

It is also enough to establish this for $z \in e^{i(-2\theta_1+\pi/2)}(1,\infty)$ as then the result follows by analytic continuation. For that we show that for $f \in \mathcal{H}_{R_0} \oplus L^2(B(0,R_1) \setminus B(0,R_0)) \subset \mathcal{H}_{\theta_j}$ there exists U holomorphic in a neighborhood $\Omega_{\theta_1,\theta_2}$ of

$$\bigcup_{\theta_1 \leq \theta \leq \theta_2} (\Gamma_\theta \setminus B(0,R_0)) \subset \mathbb{C}^n$$

such that

$$U|_{\Gamma_{\theta_j}}(x) = [(\mathcal{P}_{\varepsilon,\theta_j} - z)^{-1}\psi f](x), \quad \forall\, x \in \Gamma_{\theta_j} \setminus B(0,R_0). \tag{3.8}$$

To show the existence of U such that (3.8) holds we apply Lemma 2.1 to a modified family of deformations, which is obtained as follows. Let $\rho \in \mathcal{C}^\infty_{\rm c}((1,6);[0,1])$ be equal to 1 near $[2,4]$, and put for $T \geq 1$,

$$g_{\theta_1,\theta_2,T}(t) := g_{\theta_1}(t) + \rho(t/T)(g_{\theta_2}(t) - g_{\theta_1}(t)),$$
$$\Gamma_{\theta_1,\theta_2,T} := \{g_{\theta_1,\theta_2,T}(t)\omega : t \in [0,\infty),\ \omega \in \mathbb{S}^{n-1}\} \subset \mathbb{C}^n.$$

We can apply Lemma 2.1 to the family of totally real submanifolds interpolating between Γ_{θ_1} and $\Gamma_{\theta_1,\theta_2,T}$, $[0,1] \ni s \mapsto \Gamma_{\theta_1,(1-s)\theta_1+s\theta_2,T}$. It follows that there exists a holomorphic function U^T defined in a neighborhood of the union of these submanifolds which restricts to $u_1 := (\mathcal{P}_{\varepsilon,\theta_1} - z)^{-1}\psi f \in \mathcal{H}_{\theta_1}$. Varying T we obtain a family of functions agreeing on the intersections of their domains and that gives a holomorphic function U defined in the neighborhood $\Omega_{\theta_1,\theta_2}$.

It remains to show that U restricts to $u_2 \in \mathcal{H}_{\theta_2}$ (the equation $(\mathcal{P}_{\varepsilon,\theta_2} - z)u_2 = \psi f$ is automatically satisfied). For T large we put

$$\Omega_1(T) = \{z \in \mathbb{C}^n : T \leq |z| \leq 6T\} \cap \Gamma_{\theta_1,\theta_2,T} \supset \Gamma_{\theta_1,\theta_2,T} \setminus \Gamma_{\theta_1},$$
$$\Omega_2(T) = \{z \in \mathbb{C}^n : T/2 \leq |z| \leq 8T\} \cap \Gamma_{\theta_1,\theta_2,T}, \quad \Omega_2(T) \setminus \Omega_1(T) \subset e^{i\theta_1}\mathbb{R}^n,$$

and choose $\chi_T \in \mathcal{C}^\infty(\Omega_2(T);[0,1])$ such that $\chi_T = 1$ on $\Omega_1(T)$ with derivative bounds independent of T. We recall the following estimate from the proof of [Zw18, Lemma 3]: for $u \in \mathcal{C}^\infty(\Gamma_{\theta_1,\theta_2,T})$, $\tau > 1$,

$$|\langle(-\Delta|_{\Gamma_{\theta_1,\theta_2,T}} - i\varepsilon(x|_{\Gamma_{\theta_1,\theta_2,T}})^2 - ie^{-2i\theta_1}\tau)u, u\rangle| \geq (\|u\|^2_{L^2} + \|Du\|^2_{L^2})/C,$$

with $C > 0$ independent of τ, T, here $\langle\cdot,\cdot\rangle$ is the L^2 inner product on $\Gamma_{\theta_1,\theta_2,T}$. Writing

$$\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T} := P|_{\Gamma_{\theta_1,\theta_2,T}} - i\varepsilon(x|_{\Gamma_{\theta_1,\theta_2,T}})^2,$$

it then follows from (1.5) that

$$\langle(\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T}-(-\Delta|_{\Gamma_{\theta_1,\theta_2,T}}-i\varepsilon(x|_{\Gamma_{\theta_1,\theta_2,T}})^2))u,u\rangle=\int_{\Gamma_{\theta_1,\theta_2,T}}(g^{jk}-\delta^{jk})\partial_k u\partial_j\bar{u}+c|u|^2.$$

In view of (1.6) and (1.7), we obtain that for T sufficiently large,

$$|\langle(\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T}-ie^{-2i\theta_1}\tau)\chi_T U,\chi_T U\rangle|\geq(\|\chi_T U\|_{L^2}^2+\|D(\chi_T U)\|_{L^2}^2)/C,$$

thus $\|\chi_T U\|_{L^2}\leq C\|(\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T}-ie^{-2i\theta_1}\tau)\chi_T U\|_{L^2}$. We note that

$$(\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T}-ie^{-2i\theta_1}\tau)U^T=0\implies(\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T}-ie^{-2i\theta_1}\tau)\chi_T U=[\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T},\chi_T]U,$$

which is supported on $\Omega_2(T)\setminus\Omega_1(T)\subset\Gamma_{\theta_1}$. Hence,

$$\|1_{2T\leq|z|\leq 4T}\,u_2\|_{L^2(\Gamma_{\theta_2})}^2\leq C\|[\mathcal{P}_{\varepsilon,\theta_1,\theta_2,T},\chi_T]U\|_{L^2}^2\leq C\|1_{T/2\leq|z|\leq 8T}\,u_1\|_{H^1(\Gamma_{\theta_1})}^2.$$

We now take $T=2^j$ and sum over j, it follows that $u_2\in\mathcal{H}_{\theta_2}$. □

Lemma 3.4. *For $0\leq\theta<\theta_0$, $\varepsilon>0$, the spectrum of $\mathcal{P}_{\varepsilon,\theta}$ is independent of θ. More precisely, for any $z_0\in\mathbb{C}\setminus e^{-i\pi/4}[0,\infty)$ we have*

$$m_{\varepsilon,\theta}(z_0):=\operatorname{rank}\oint_{z_0}(\mathcal{P}_{\varepsilon,\theta}-z)^{-1}dz=\operatorname{rank}\oint_{z_0}(P_\varepsilon-z)^{-1}\,dz,\tag{3.9}$$

where the integral is over a positively oriented circle enclosing z_0 and containing no poles other than possibly z_0.

Proof. Lemma 3.2 shows that

$$\Pi_{\varepsilon,\theta}(z_0):=-\frac{1}{2\pi i}\oint_{z_0}(\mathcal{P}_{\varepsilon,\theta}-z)^{-1}dz,\tag{3.10}$$

is a finite rank projection which maps $\mathcal{H}_\theta$ to the generalized eigenspace of $\mathcal{P}_{\varepsilon,\theta}$ at z_0. In view of Lemma 3.3, it suffices to show that for each $0\leq\theta<\theta_0$,

$$\operatorname{rank}\Pi_{\varepsilon,\theta}(z_0)=\operatorname{rank}\psi\Pi_{\varepsilon,\theta}(z_0)\psi.$$

First we show that $\operatorname{rank}\Pi_{\varepsilon,\theta}(z_0)=\operatorname{rank}\Pi_{\varepsilon,\theta}(z_0)\psi$, which is equivalent to show that $\operatorname{rank}\psi\Pi_{\varepsilon,\theta}(z_0)^*=\operatorname{rank}\Pi_{\varepsilon,\theta}(z_0)^*$, since the adjoint of a finite rank operator is of finite rank with the same rank. For that we shall argue by contradiction. Suppose that $\operatorname{rank}\psi\Pi_{\varepsilon,\theta}(z_0)^*<\operatorname{rank}\Pi_{\varepsilon,\theta}(z_0)^*$, there would exist $0\neq\tilde{v}\in\operatorname{Ran}\Pi_{\varepsilon,\theta}(z_0)^*$ satisfying $\psi\tilde{v}=0$. Note that $\Pi_{\varepsilon,\theta}(z_0)^*$ is also a projection of the form (3.10) except that $\mathcal{P}_{\varepsilon,\theta}^*$ and $\bar{z}_0$ replace $\mathcal{P}_{\varepsilon,\theta}$ and z_0 there, we may assume

$$(\mathcal{P}_{\varepsilon,\theta}^*-\bar{z}_0)^k\tilde{v}=0,\quad\tilde{u}:=(\mathcal{P}_{\varepsilon,\theta}^*-\bar{z}_0)^{k-1}\tilde{v}\neq 0,\quad\text{for some }k\geq 1.$$

But that would mean that $\tilde{u}$ can be identified with an element of $H^2(\Gamma_\theta)$ satisfying

$$(Q_{\varepsilon,\theta}^*-\bar{z}_0)\tilde{u}=0,\quad\tilde{u}|_{B(0,R_0)}\equiv 0,\quad Q_{\varepsilon,\theta}:=Q_\theta-i\varepsilon(1-\chi(x_\theta))x_\theta^2.$$

Since $Q_{\varepsilon,\theta}^*$ is elliptic, unique continuation results for second order elliptic differential equations – see Hörmander [HöIII, Chapter 17] show that $\tilde{u}\equiv 0$, thus a contradiction.

It remains to show that $\operatorname{rank} \psi\Pi_{\varepsilon,\theta}(z_0)\psi = \operatorname{rank} \Pi_{\varepsilon,\theta}(z_0)\psi$. Otherwise there would exist solutions $v \in \widehat{\mathcal{D}}_\theta$ to $(\mathcal{P}_{\varepsilon,\theta} - z_0)^\ell v = 0$, $u := (\mathcal{P}_{\varepsilon,\theta} - z_0)^{\ell-1} v \neq 0$ with $\psi v = 0$. It follows that u can be identified with an element of $H^2(\Gamma_\theta)$ satisfying

$$(Q_{\varepsilon,\theta} - z_0)u = 0, \quad u|_{B(0,R_0)} \equiv 0.$$

Again by the unique continuation results for second order elliptic differential equations, we obtain that $u \equiv 0$, thus a contradiction. $\square$

The next lemma shows that the spectrum of $\mathcal{P}_{\varepsilon,\theta}$ must stay close to the spectrum of $\mathcal{P}_\theta$ when ε is sufficiently small:

Lemma 3.5. *Suppose that $0 \le \theta < \theta_0$ and that $\Omega \Subset \{z : -2\theta < \arg z < 3\pi/2 + 2\theta\}$ is disjoint with $\operatorname{Spec}(\mathcal{P}_\theta)$, then there exist $\varepsilon_0 = \varepsilon_0(\Omega)$ and $C = C(\Omega)$ such that, uniformly in $0 < \varepsilon < \varepsilon_0$, $\operatorname{Spec}(\mathcal{P}_{\varepsilon,\theta}) \cap \Omega = \emptyset$ and*

$$\|(\mathcal{P}_{\varepsilon,\theta} - z)^{-1}\|_{\mathcal{H}_\theta \to \mathcal{D}_\theta} \le C, \quad z \in \Omega.$$

Proof. We follow closely the proof of [Zw18, Lemma 5] except that $\mathcal{P}_\theta$ replaces $-\Delta_\theta$ there. Let $\chi_j \in \mathcal{C}_c^\infty([0,\infty);[0,1])$ be equal to 1 on $[0, R_0]$ and satisfy $\chi_j = 1$ near $\operatorname{supp} \chi_{j-1}$, $j = 1, 2$. Parametrizing Γ_θ by $f_\theta : [0,\infty) \times \mathbb{S}^{n-1} \ni (t,\omega) \mapsto g_\theta(t)\omega \in \Gamma_\theta$, we define functions $\chi_j^h \in \mathcal{C}_c^\infty(\Gamma_\theta)$ by

$$\chi_j^h(g_\theta(t)\omega) := \chi_j(th), \quad 0 < h \le 1.$$

For $z \in \Omega$ we put

$$E_{\varepsilon,\theta}^h(z) := \chi_2^h(\mathcal{P}_\theta - z)^{-1}\chi_1^h + (1 - \chi_0^h)(H_{\varepsilon,\theta} - z)^{-1}(1 - \chi_1^h),$$

so that $(\mathcal{P}_{\varepsilon,\theta} - z)E_{\varepsilon,\theta}^h(z) = I + K_{\varepsilon,\theta}^h(z)$, where

$$\begin{aligned} K_{\varepsilon,\theta}^h(z) := &- i\varepsilon(1-\chi)x_\theta^2\chi_2^h(\mathcal{P}_\theta - z)^{-1}\chi_1^h + [\mathcal{P}_\theta, \chi_2^h](\mathcal{P}_\theta - z)^{-1}\chi_1^h \\ &+ (\mathcal{P}_{\varepsilon,\theta} - H_{\varepsilon,\theta})(1-\chi_0^h)(H_{\varepsilon,\theta} - z)^{-1}(1-\chi_1^h) \\ &- [\mathcal{P}_\theta, \chi_0^h](1-\chi_0^h)(H_{\varepsilon,\theta} - z)^{-1}(1-\chi_1^h). \end{aligned}$$

Using (2.5) and (3.4) we see that for h small enough,

$$\|(\mathcal{P}_{\varepsilon,\theta} - H_{\varepsilon,\theta})(1-\chi_0^h)(H_{\varepsilon,\theta} - z)^{-1}(1-\chi_1^h)\|_{L^2(\Gamma_\theta)\to L^2(\Gamma_\theta)} < 1/4.$$

Since $[Q_\theta, \chi_j^h] = O(h) : H^1(\Gamma_\theta) \to L^2(\Gamma_\theta)$ and $x_\theta^2\chi_2^h = O(h^{-2}) : L^2(\Gamma_\theta) \to L^2(\Gamma_\theta)$, we can first choose h sufficiently small then there exists $\varepsilon_0 = \varepsilon_0(h,\Omega)$ such that for all $\varepsilon < \varepsilon_0(h,\Omega)$ and $z \in \Omega$, $\|K_{\varepsilon,\theta}^h(z)\|_{\mathcal{H}_\theta\to\mathcal{H}_\theta} < 1/2$, thus $I + K_{\varepsilon,\theta}^h(z)$ has a uniformly bounded inverse and $(\mathcal{P}_{\varepsilon,\theta} - z)^{-1} = E_{\varepsilon,\theta}^h(z)(I + K_{\varepsilon,\theta}^h(z))^{-1}$ exists. It follows from (3.4) that there exists $C = C(\Omega)$ independent of ε such that for $z \in \Omega$, $\|E_{\varepsilon,\theta}^h(z)\|_{\mathcal{H}_\theta\to\mathcal{D}_\theta} \le C$, which completes the proof. $\square$

4. The obstacle problem and the Dirichlet-to-Neumann operator

In the black box case we cannot use the strategy of [Zw18] which covers the case $P = -\Delta + V$, $V \in L^\infty_{\mathrm{comp}}$. Instead we introduce an artificial obstacle to separate the abstract black box from the differential operator outside. By an *obstacle* we mean a bounded open set $\mathcal{O}$ with smooth boundary as in §2.2. Suppose that $\mathcal{O}$ contains $\overline{B(0,R_0)}$ and that χ in (1.8) be equal to 1 near $\overline{\mathcal{O}}$. Let $\nu(x)$ be the Euclidean normal vector of $\partial\mathcal{O}$ at x pointing into $\mathcal{O}$, we put

$$\nu_g(x) := (g^{jk}(x))_{n\times n} \cdot \nu(x), \quad x \in \partial\mathcal{O}. \tag{4.1}$$

First we introduce the interior Dirichlet-to-Neumann operator of P:

$$\mathcal{N}_P^{\mathrm{in}}(z)\varphi := \frac{\partial u}{\partial \nu_g}, \quad \text{where } u \text{ solves the problem } \begin{array}{c} (P-z)u = 0 \text{ in } \mathcal{O} \\ u = \varphi \text{ on } \partial\mathcal{O} \end{array}. \tag{4.2}$$

$\mathcal{N}_P^{\mathrm{in}}(z)$ is well-defined once we establish the existence and uniqueness of the solution u to the boundary-value problem in (4.2). This can be done if z is not an eigenvalue of the operator $P^{\mathcal{O}}$ introduced in §2.2. Indeed, we set $E^{\mathrm{in}} : H^{3/2}(\partial\mathcal{O}) \to H^2(\mathcal{O})$ as a linear bounded extension operator such that $E^{\mathrm{in}}\varphi|_{\partial\mathcal{O}} = \varphi$ and $\operatorname{supp} E^{\mathrm{in}}\varphi \subset \overline{\mathcal{O}} \setminus B(0,R_0)$ for any φ. Then for $z \notin \operatorname{Spec}(P^{\mathcal{O}})$, $u = E^{\mathrm{in}}\varphi - (P^{\mathcal{O}} - z)^{-1}(Q-z)E^{\mathrm{in}}\varphi$ is the unique solution to (4.2), we obtain that

$$\mathcal{N}_P^{\mathrm{in}}(z)\varphi = \partial_{\nu_g}(E^{\mathrm{in}}\varphi - (P^{\mathcal{O}} - z)^{-1}(Q-z)E^{\mathrm{in}}\varphi), \tag{4.3}$$

Hence $z \mapsto \mathcal{N}_P^{\mathrm{in}}(z) : H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$ is a meromorphic family of operators on $\mathbb{C}$ with poles contained in $\operatorname{Spec}(P^{\mathcal{O}})$.

Similarly, we can define the exterior Dirichlet-to-Neumann operator of $\mathcal{P}_{\varepsilon,\theta}$ for every $0 \le \theta < \theta_0$ and $\varepsilon \ge 0$:

$$\mathcal{N}_{\varepsilon,\theta}^{\mathrm{out}}(z)\varphi := \frac{\partial u}{\partial \nu_g}, \quad \text{where } u \text{ solves the problem } \begin{array}{c} (Q_{\varepsilon,\theta}-z)u = 0 \text{ in } \Gamma_\theta \setminus \mathcal{O} \\ u = \varphi \text{ on } \partial\mathcal{O} \end{array}. \tag{4.4}$$

To show the well-definedness of $\mathcal{N}_{\varepsilon,\theta}^{\mathrm{out}}(z)$, we introduce the restriction of $Q_{\varepsilon,\theta}$ to $\Gamma_\theta \setminus \mathcal{O}$ with Dirichlet boundary condition as follows:

$$\begin{gathered} Q_\theta^{\mathcal{O}} : H^2(\Gamma_\theta \setminus \mathcal{O}) \cap H_0^1(\Gamma_\theta \setminus \mathcal{O}) \to L^2(\Gamma_\theta \setminus \mathcal{O}), \quad Q_\theta^{\mathcal{O}} u := Q_\theta u, \\ Q_{\varepsilon,\theta}^{\mathcal{O}} := Q_\theta^{\mathcal{O}} - i\varepsilon(1-\chi)x_\theta^2 \quad \text{with domain } \mathcal{D}(Q_\theta^{\mathcal{O}}) \cap |x_\theta|^{-2}L^2(\Gamma_\theta \setminus \mathcal{O}). \end{gathered} \tag{4.5}$$

Since $Q_\theta^{\mathcal{O}}$ and $Q_{\varepsilon,\theta}^{\mathcal{O}}$ can also be viewed as black box perturbations of $-\Delta_\theta$ and $H_{\varepsilon,\theta}$ respectively, we conclude from Lemma 2.2 and Lemma 3.2 that $Q_{\varepsilon,\theta}^{\mathcal{O}} - z$, $\varepsilon \ge 0$ is a Fredholm operator of index 0 for $-2\theta < \arg z < 3\pi/2 + 2\theta$. We claim that $\mathcal{N}_{\varepsilon,\theta}^{\mathrm{out}}(z)$ is well defined if $z \notin \operatorname{Spec}(Q_{\varepsilon,\theta}^{\mathcal{O}})$. For that let $E^{\mathrm{out}} : H^{3/2}(\partial\mathcal{O}) \to H^2(\Gamma_\theta \setminus \mathcal{O})$ be a linear bounded extension operator with $E^{\mathrm{out}}\varphi|_{\partial\mathcal{O}} = \varphi$ and $\operatorname{supp} E^{\mathrm{out}}\varphi \Subset \Gamma_\theta \setminus \mathcal{O}$, then

$$\mathcal{N}_{\varepsilon,\theta}^{\mathrm{out}}(z)\varphi = \partial_{\nu_g}(E^{\mathrm{out}}\varphi - (Q_{\varepsilon,\theta}^{\mathcal{O}} - z)^{-1}(Q_{\varepsilon,\theta} - z)E^{\mathrm{out}}\varphi). \tag{4.6}$$

It follows that $z \mapsto \mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z) : H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$ is a meromorphic family of operators in the region $-2\theta < \arg z < 3\pi/2 + 2\theta$, with poles contained in $\operatorname{Spec}(Q^{\mathcal{O}}_{\varepsilon,\theta})$.

Now we put

$$\mathcal{N}_{\varepsilon,\theta}(z) := \mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z) - \mathcal{N}^{\text{in}}_{P}(z). \tag{4.7}$$

Lemma 4.1. *Suppose that $0 \le \theta < \theta_0$, $\varepsilon \ge 0$ and that $-2\theta < \arg z < 3\pi/2 + 2\theta$ with $z \notin \operatorname{Spec}(P^{\mathcal{O}}) \cup \operatorname{Spec}(Q^{\mathcal{O}}_{\varepsilon,\theta})$, then $\mathcal{N}_{\varepsilon,\theta}(z) : H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$ is a Fredholm operator of index 0.*

Proof. Let $Q^{\mathcal{O}}_{\text{in}}$ and $\mathcal{N}^{\text{in}}_{Q}(z)$ be the the reference operator and the interior Dirichlet-to-Neumann operator associated with Q, defined as in (2.12) and (4.2) respectively except that Q replaces P there. Choosing $z_0 \notin \operatorname{Spec}(Q^{\mathcal{O}}_{\text{in}}) \cup \operatorname{Spec}(Q^{\mathcal{O}}_{\varepsilon,\theta}) \cup \operatorname{Spec}(Q_{\varepsilon,\theta})$, we claim that

$$\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0) - \mathcal{N}^{\text{in}}_{Q}(z_0) : H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O}) \quad \text{is invertible.} \tag{4.8}$$

To show injectivity, we argue by contradiction. Suppose that $0 \ne \varphi \in H^{3/2}(\partial\mathcal{O})$ satisfies $\mathcal{N}^{\text{out}}_{\varepsilon,\theta}(z_0)\varphi = \mathcal{N}^{\text{in}}_{Q}(z_0)\varphi$, it follows from (4.2) and (4.4) that there exist $u_1 \in H^2(\mathcal{O})$, $u_2 \in H^2(\Gamma_\theta \setminus \mathcal{O})$ ($|x_\theta|^2 u_2 \in L^2(\Gamma_\theta \setminus \mathcal{O})$ when $\varepsilon > 0$) such that

$$u_1 \text{ solves } \begin{array}{c} (Q - z_0)u_1 = 0 \text{ in } \mathcal{O} \\ u_1 = \varphi \text{ on } \partial\mathcal{O} \end{array}, \text{ and } u_2 \text{ solves } \begin{array}{c} (Q_{\varepsilon,\theta} - z_0)u_2 = 0 \text{ in } \Gamma_\theta \setminus \mathcal{O} \\ u_2 = \varphi \text{ on } \partial\mathcal{O} \end{array}, \tag{4.9}$$

and that $\partial_{\nu_g} u_1 = \partial_{\nu_g} u_2$. Let $u = 1_{\mathcal{O}}\, u_1 + 1_{\Gamma_\theta \setminus \mathcal{O}}\, u_2$, we aim to show that $u \in H^2(\Gamma_\theta)$. For that it suffices to show the regularity of u near $\partial\mathcal{O}$. For any $x_0 \in \partial\mathcal{O}$, we choose $B_{x_0} := B(x_0, r) \subset B(0, R_1)$ such that $Q_{\varepsilon,\theta} = Q$ in B_{x_0} and put $v \in \mathcal{C}^\infty_{\mathrm{c}}(B_{x_0})$. Then we integrate by parts to obtain:

$$\begin{aligned}
&\int_{B_{x_0}} \left(\sum_{j,k=1}^{n} g^{jk} \partial_{x_k} u\, \partial_{x_j} v + cuv \right) dx \\
&= \int_{B_{x_0} \cap \mathcal{O}} \left(\sum_{j,k=1}^{n} g^{jk} \partial_{x_k} u_1 \partial_{x_j} v + c u_1 v \right) dx + \int_{B_{x_0} \setminus \mathcal{O}} \left(\sum_{j,k=1}^{n} g^{jk} \partial_{x_k} u_2 \partial_{x_j} v + c u_2 v \right) dx \\
&= \int_{B_{x_0} \cap \mathcal{O}} v\, Q u_1 dx - \int_{\partial\mathcal{O} \cap B_{x_0}} v\, \partial_{\nu_g} u_1 dS(x) + \int_{B_{x_0} \setminus \mathcal{O}} v\, Q u_2 dx + \int_{\partial\mathcal{O} \cap B_{x_0}} v\, \partial_{\nu_g} u_1 dS(x) \\
&= \int_{B_{x_0} \cap \mathcal{O}} z_0 u_1 v\, dx + \int_{B_{x_0} \setminus \mathcal{O}} z_0 u_2 v\, dx = \int_{B_{x_0}} z_0 u v\, dx.
\end{aligned}$$

Hence u is a weak solution of $(Q - z_0)u = 0$ in B_{x_0}, the regularity results for second order elliptic differential equations show that u is H^2 near x_0, thus $u \in H^2(\Gamma_\theta)$. It then follows from (4.9) that u solves the equation $(Q_{\varepsilon,\theta} - z_0)u = 0$, thus $z_0 \in \operatorname{Spec}(Q_{\varepsilon,\theta})$, which gives a contradiction.

To show surjectivity, we first choose a linear bounded operator $L_g : H^{1/2}(\partial\mathcal{O}) \to H^2(\mathcal{O})$ satisfying the following:

$$\begin{gathered} L_g\tilde{\varphi} := v, \quad \text{where } v \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}) \text{ satisfies} \\ \operatorname{supp} v \subset \overline{\mathcal{O}} \setminus B(0,R_0) \text{ and } \partial_{\nu_g} v = \tilde{\varphi}, \quad \tilde{\varphi} \in H^{1/2}(\partial\mathcal{O}). \end{gathered} \tag{4.10}$$

For any $\tilde{\varphi} \in H^{1/2}(\partial\mathcal{O})$, let $v := L_g\tilde{\varphi}$, $f := (Q_{\text{in}}^{\mathcal{O}} - z_0)v \in L^2(\mathcal{O})$ and we put

$$u := (Q_{\varepsilon,\theta} - z_0)^{-1}\imath f \quad \text{and} \quad \varphi := u|_{\partial\mathcal{O}} \in H^{3/2}(\mathcal{O}),$$

where $\imath : L^2(\mathcal{O}) \hookrightarrow L^2(\Gamma_\theta)$ denotes the extension by zero. Then $u_1 := 1_{\mathcal{O}}u - v$ solves the boundary value problem $(Q - z_0)u_1 = 0$ in $\mathcal{O}$, $u_1 = \varphi$ on $\partial\mathcal{O}$; $u_2 := 1_{\Gamma_\theta\setminus\mathcal{O}}\, u$ solves $(Q_{\varepsilon,\theta} - z_0)u_2 = 0$ in $\Gamma_\theta \setminus \mathcal{O}$, $u_2 = \varphi$ on $\partial\mathcal{O}$. Hence we have

$$\mathcal{N}_{\varepsilon,\theta}^{\text{out}}(z_0)\varphi - \mathcal{N}_Q^{\text{in}}(z_0)\varphi = \partial_{\nu_g} 1_{\Gamma_\theta\setminus\mathcal{O}}\, u - \partial_{\nu_g}(1_{\mathcal{O}}u - v) = \partial_{\nu_g} v = \tilde{\varphi}.$$

In view of (4.8), we now show that $\mathcal{N}_{\varepsilon,\theta}^{\text{out}}(z) - \mathcal{N}_{\varepsilon,\theta}^{\text{out}}(z_0)$ and $\mathcal{N}_P^{\text{in}}(z) - \mathcal{N}_Q^{\text{in}}(z_0)$ are compact: $H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$. Using (4.6) we have for any $\varphi \in H^{3/2}(\mathcal{O})$,

$$\begin{aligned} &\mathcal{N}_{\varepsilon,\theta}^{\text{out}}(z)\varphi - \mathcal{N}_{\varepsilon,\theta}^{\text{out}}(z_0)\varphi \\ &= \partial_{\nu_g}((Q_{\varepsilon,\theta}^{\mathcal{O}} - z_0)^{-1}(Q_{\varepsilon,\theta} - z_0) - (Q_{\varepsilon,\theta}^{\mathcal{O}} - z)^{-1}(Q_{\varepsilon,\theta} - z))E^{\text{out}}\varphi \\ &= (z - z_0)\partial_{\nu_g}(Q_{\varepsilon,\theta}^{\mathcal{O}} - z_0)^{-1}(I - (Q_{\varepsilon,\theta}^{\mathcal{O}} - z)^{-1}(Q_{\varepsilon,\theta} - z))E^{\text{out}}\varphi \in H^{5/2}(\partial\mathcal{O}), \end{aligned}$$

thus $\mathcal{N}_{\varepsilon,\theta}^{\text{out}}(z) - \mathcal{N}_{\varepsilon,\theta}^{\text{out}}(z_0) : H^{3/2}(\partial\mathcal{O}) \to H^{5/2}(\partial\mathcal{O}) \subset H^{1/2}(\partial\mathcal{O})$ is compact since the last inclusion map is compact. It remains to show that $\mathcal{N}_P^{\text{in}}(z) - \mathcal{N}_Q^{\text{in}}(z_0)$ is compact: $H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$. Let $\psi \in \mathcal{C}_c^\infty(\mathcal{O})$ be equal to 1 near $\overline{B(0,R_0)}$, $\varphi \in H^{1/2}(\mathcal{O})$, there exist u and v satisfying:

$$\begin{aligned} (P - z)u &= 0 \text{ in } \mathcal{O} \\ u &= \varphi \text{ on } \partial\mathcal{O} \end{aligned} \quad \text{and} \quad \begin{aligned} (Q - z_0)v &= 0 \text{ in } \mathcal{O} \\ v &= \varphi \text{ on } \partial\mathcal{O} \end{aligned},$$

recalling (2.11) that $(1-\psi)u \in H^2(\mathcal{O})$, thus we have

$$(\mathcal{N}_P^{\text{in}}(z) - \mathcal{N}_Q^{\text{in}}(z_0))\varphi = \partial_{\nu_g}((1-\psi)u - (1-\psi)v).$$

Using (1.5) we can show that $(1-\psi)u - (1-\psi)v \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ satisfies:

$$\begin{aligned} Q((1-\psi)u - (1-\psi)v) &= (1-\psi)Pu - [P,\psi]u - (1-\psi)Qv + [Q,\psi]v \\ &= z(1-\psi)u - z_0(1-\psi)v - [P,\psi]u + [Q,\psi]v \in H^1(\mathcal{O}), \end{aligned}$$

then we conclude from the regularity results for second order elliptic differential equations that $(1-\psi)u - (1-\psi)v \in H^3(\mathcal{O})$, thus $(\mathcal{N}_P^{\text{in}}(z) - \mathcal{N}_Q^{\text{in}}(z_0))\varphi \in H^{3/2}(\partial\mathcal{O})$. Therefore, $\mathcal{N}_P^{\text{in}}(z) - \mathcal{N}_Q^{\text{in}}(z_0) : H^{3/2}(\partial\mathcal{O}) \to H^{3/2}(\partial\mathcal{O}) \subset H^{1/2}(\partial\mathcal{O})$ is compact.

So far we have shown that there exists a compact operator $K(z) : H^{3/2}(\partial\mathcal{O}) \to H^{1/2}(\partial\mathcal{O})$ such that $\mathcal{N}_{\varepsilon,\theta}(z) = \mathcal{N}_{\varepsilon,\theta}^{\text{out}}(z_0) - \mathcal{N}_Q^{\text{in}}(z_0) + K(z)$. Using (4.8) we can write

$$\mathcal{N}_{\varepsilon,\theta}(z) = (\mathcal{N}_{\varepsilon,\theta}^{\text{out}}(z_0) - \mathcal{N}_Q^{\text{in}}(z_0))(I + (\mathcal{N}_{\varepsilon,\theta}^{\text{out}}(z_0) - \mathcal{N}_Q^{\text{in}}(z_0))^{-1}K(z)),$$

i.e. it is a product of an invertible operator and a Fredholm operator of index 0, thus $\mathcal{N}_{\varepsilon,\theta}(z)$ is also a Fredholm operator of index 0. $\square$

Remark: The compactness of $\mathcal{N}^{\rm out}_{\varepsilon,\theta}(z) - \mathcal{N}^{\rm out}_{\varepsilon,\theta}(z_0)$ and $\mathcal{N}^{\rm in}_{P}(z) - \mathcal{N}^{\rm in}_{Q}(z_0)$ can also be proved using the facts that the principal symbols of $\mathcal{N}^{\rm out}_{\varepsilon,\theta}(z)$ and $\mathcal{N}^{\rm out}_{\varepsilon,\theta}(z_0)$ are identical, same for $\mathcal{N}^{\rm in}_{P}(z)$ and $\mathcal{N}^{\rm in}_{Q}(z_0)$ – see for instance Lee–Uhlmann [LeUh89] for a detailed account.

In order to work on a single Hilbert space, we introduce

$$\widehat{\mathcal{N}}_{\varepsilon,\theta}(z) := \langle D_{\partial\mathcal{O}}\rangle^{-1}\mathcal{N}_{\varepsilon,\theta}(z) : H^{3/2}(\partial\mathcal{O}) \to H^{3/2}(\partial\mathcal{O}), \tag{4.11}$$

where $\langle D_{\partial\mathcal{O}}\rangle = (1-\Delta_{\partial\mathcal{O}})^{1/2}$ is the standard isomorphism between Sobolev spaces $H^s(\partial\mathcal{O})$ and $H^{s-1}(\partial\mathcal{O})$. Now we are ready to state the main results of this section:

Lemma 4.2. *Suppose that* $0 \le \theta < \theta_0$, $\varepsilon \ge 0$ *and that* $\Omega \Subset \{z : -2\theta < \arg z < 3\pi/2 + 2\theta\}$ *is disjoint from* $\operatorname{Spec}(P^{\mathcal{O}}) \cup \operatorname{Spec}(Q^{\mathcal{O}}_{\varepsilon,\theta})$,

$$z \mapsto \widehat{\mathcal{N}}_{\varepsilon,\theta}(z)^{-1}, \quad z \in \Omega,$$

is a meromorphic family of operators on $H^{3/2}(\partial\mathcal{O})$ *with poles of finite rank. Moreover,*

$$n_{\varepsilon,\theta}(z) := \frac{1}{2\pi i}\operatorname{tr}\oint_z \widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1}\partial_w\widehat{\mathcal{N}}_{\varepsilon,\theta}(w)\,dw = m_{\varepsilon,\theta}(z), \tag{4.12}$$

where the integral is over a positively oriented circle enclosing z *and containing no poles other than possibly* z *and* $m_{\varepsilon,\theta}(z)$ *is given by* (3.9) *(and by* (2.9) *when* $\varepsilon = 0$*).*

Proof. 1. Suppose that $z \in \Omega$ is an eigenvalue of $\mathcal{P}_{\varepsilon,\theta}$, we choose $u \in \ker(\mathcal{P}_{\varepsilon,\theta} - z)$ and let $\varphi = u|_{\partial\mathcal{O}}$, then $\mathcal{N}^{\rm out}_{\varepsilon,\theta}(z)\varphi - \mathcal{N}^{\rm in}_{P}(z)\varphi = \partial_{\nu_g}u - \partial_{\nu_g}u = 0$. Note that $\varphi \neq 0$ since $z \notin \operatorname{Spec}(P^{\mathcal{O}})$, thus $\ker\widehat{\mathcal{N}}_{\varepsilon,\theta}(z) \neq \{0\}$. On the other hand, suppose that $0 \neq \varphi \in \ker\widehat{\mathcal{N}}_{\varepsilon,\theta}(z)$, the same arguments as in the proof of Lemma 4.1 show that $z \in \operatorname{Spec}(\mathcal{P}_{\varepsilon,\theta})$. Hence

$$z \in \operatorname{Spec}(\mathcal{P}_{\varepsilon,\theta}) \Longleftrightarrow \ker\widehat{\mathcal{N}}_{\varepsilon,\theta}(z) \neq \{0\}, \tag{4.13}$$

and we conclude from Lemma 4.1 that $\widehat{\mathcal{N}}_{\varepsilon,\theta}(z)$ is invertible for $z \in \Omega \setminus \operatorname{Spec}(\mathcal{P}_{\varepsilon,\theta})$. Analytic Fredholm theory then shows that $\Omega \ni z \mapsto \widehat{\mathcal{N}}_{\varepsilon,\theta}(z)^{-1}$ is a meromorphic family of operators on $H^{3/2}(\partial\mathcal{O})$ with poles of finite rank.

2. Since (4.13) proves (4.12) in the case $m_{\varepsilon,\theta}(z) = 0$, we now assume that $m_{\varepsilon,\theta}(z) = M \ge 1$, and that $\mathcal{P}_{\varepsilon,\theta}$ has exactly one eigenvalue z in $D(z, 2r) := \{\zeta \in \mathbb{C}, |\zeta - z| < 2r\}$. We note that z is not a compactly supported embedded eigenvalue of P, by which we mean an eigenvalue admitting a compactly supported eigenfunction – see (5.17). This is because if $(P-z)u = 0$ for some $0 \neq u \in \mathcal{D}_{\rm comp}$, then u must vanish identically outside $B(0, R_0)$ by unique continuation results for second order elliptic differential equations, thus $u \in \mathcal{D}^{\mathcal{O}}$. It follows that $z \in \operatorname{Spec}(P^{\mathcal{O}})$ which contradicts

the assumption $\Omega \cap \mathrm{Spec}(P^{\mathcal{O}}) = \emptyset$. Then we claim that for any $\delta > 0$ there exists $V \in \mathcal{C}^\infty(\mathcal{O} \setminus B(0, R_0); \mathbb{R})$ with $\|V\|_\infty < \delta$ such that

$$\mathrm{rank} \int_{\partial D(z,r)} (\mathcal{P}_{\varepsilon,\theta} + V - w)^{-1} dw = M,$$

and that the eigenvalues of $\mathcal{P}_{\varepsilon,\theta} + V$ in $D(z, r)$ are all simple. This follows from the results of Klopp–Zworski [KlZw95] (see also [DyZw19, Theorem 4.39]) and we omit the proof here. Replacing P by $P + V$ in (4.2), we can define $\widehat{\mathcal{N}}^V_{\varepsilon,\theta}$ for $\mathcal{P}_{\varepsilon,\theta} + V$ as in (4.7) and (4.11). Note that $\widehat{\mathcal{N}}_{\varepsilon,\theta}$ has no kernel except at z in $D(z, 2r)$ by (4.13), using (4.3) we can choose δ small enough such that for $\|V\|_\infty < \delta$,

$$\|\widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1}(\widehat{\mathcal{N}}_{\varepsilon,\theta}(w) - \widehat{\mathcal{N}}^V_{\varepsilon,\theta}(w))\|_{H^{3/2}(\mathcal{O})\to H^{3/2}(\mathcal{O})} < 1, \quad \forall w \in \partial D(z, r).$$

It then follows from the Gohberg–Sigal–Rouché theorem (see Gohberg–Sigal [GoSi71] and [DyZw19, Appendix C]) that

$$\frac{1}{2\pi i} \mathrm{tr} \int_{\partial D(z,r)} \mathcal{N}^V_{\varepsilon,\theta}(w)^{-1} \partial_w \mathcal{N}^V_{\varepsilon,\theta}(w)\, dw = n_{\varepsilon,\theta}(z).$$

Hence it is enough to prove (4.12) in the case $m_{\varepsilon,\theta}(z) = 1$ with $\mathcal{P}_{\varepsilon,\theta}$ replaced by $\mathcal{P}_{\varepsilon,\theta}+V$.

3. Now we assume that $m_{\varepsilon,\theta}(z) = 1$. In view of (4.13), $\widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1}$ has a pole at z, it remains to show that z is a simple pole. For any w near z and $\tilde{\varphi} \in H^{1/2}(\partial\mathcal{O})$, we recall (4.10) that $L_g\tilde{\varphi} \in \mathcal{D}^{\mathcal{O}}$, then $(P^{\mathcal{O}} - w)L_g\tilde{\varphi} \in \mathcal{H}^{\mathcal{O}}$. Now we put

$$u := (\mathcal{P}_{\varepsilon,\theta} - w)^{-1}\imath(P^{\mathcal{O}} - w)L_g\tilde{\varphi}, \quad \varphi := u|_{\partial\mathcal{O}},$$

where $\imath : \mathcal{H}^{\mathcal{O}} \hookrightarrow \mathcal{H}_\theta$ is the extension by zero. Following the arguments in the proof of Lemma 4.1 while P replacing Q there, we can show that $\mathcal{N}_{\varepsilon,\theta}(w)\varphi = \tilde{\varphi}$, thus

$$\widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1}\tilde{\varphi} = ((\mathcal{P}_{\varepsilon,\theta} - w)^{-1}\imath(P^{\mathcal{O}} - w)L_g(\langle D_{\partial\mathcal{O}}\rangle\tilde{\varphi}))|_{\partial\mathcal{O}}, \quad \forall\tilde{\varphi} \in H^{3/2}(\partial\mathcal{O}).$$

Since z is a simple pole of $w \mapsto (\mathcal{P}_{\varepsilon,\theta} - w)^{-1}$ by our assumptions, it follows from the expression above that z must be a simple pole of $w \mapsto \widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1}$. □

5. Deformation of obstacles

We have shown that the eigenvalues of $\mathcal{P}_{\varepsilon,\theta}$, $\varepsilon \geq 0$, can be identified with the poles of $z \mapsto \mathcal{N}_{\varepsilon,\theta}(z)^{-1}$. One problem of this characterization is that $\mathcal{N}_{\varepsilon,\theta}(z)$ can only be defined away from $\mathrm{Spec}(P^{\mathcal{O}})$ and $\mathrm{Spec}(Q^{\mathcal{O}}_{\varepsilon,\theta})$. In this section we will show that the spectrum of $P^{\mathcal{O}}$ and $Q^{\mathcal{O}}_\theta$ can be moved by deforming the obstacle $\mathcal{O}$ while we always assume that $\overline{B(0, R_0)} \subset \mathcal{O} \subset B(0, R_1)$. Hence for any resonance z_0 of P, we can always assume that $\mathcal{N}_\theta(z)$ is well-defined in some neighborhood of z_0 by selecting a proper obstacle.

To describe the deformations of obstacles, we follow Pereira [Pe04] and introduce a set of smooth mappings which deforms the obstacle $\mathcal{O}$:

$$\mathrm{Diff}(\mathcal{O}) := \left\{ \begin{matrix} \Phi \in \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ is a diffeomorphism : } \Phi(\partial\mathcal{O}) = \partial\,\Phi(\mathcal{O}), \\ \Phi(x) = x, \quad \text{for all } |x| \le R_0 \text{ or } |x| \ge R_1. \end{matrix} \right\} \tag{5.1}$$

We note that $\Phi \in \mathrm{Diff}(\mathcal{O})$ only deforms the region $\{x \in \mathbb{R}^n : R_0 < |x| < R_1\}$, then it also defines a diffeomorphism of Γ_θ, $0 \le \theta < \theta_0$. The pullback Φ^* gives an isomorphism between $L^2(\Gamma_\theta \setminus \Phi(\mathcal{O}))$ and $L^2(\Gamma_\theta \setminus \mathcal{O})$, which also restricts to an isomorphism between $\mathcal{D}(Q_\theta^{\Phi(\mathcal{O})})$ and $\mathcal{D}(Q_\theta^{\mathcal{O}})$ given in (4.5) since it preserves the Dirichlet boundary condition. Hence we can define

$$Q_{\theta,\Phi}^{\mathcal{O}} := \Phi^* Q_\theta^{\Phi(\mathcal{O})}(\Phi^*)^{-1}, \quad \text{with } \mathcal{D}(Q_{\theta,\Phi}^{\mathcal{O}}) = \mathcal{D}(Q_\theta^{\mathcal{O}}), \tag{5.2}$$

which is considered as the deformed operator of $Q_\theta^{\mathcal{O}}$ under the deformation Φ. The Fredholm properties of $Q_\theta^{\Phi(\mathcal{O})} - z$ immediately show that $Q_{\theta,\Phi}^{\mathcal{O}} - z$ is a Fredholm operator of index 0 for $-2\theta < \arg z < 3\pi/2 + 2\theta$, and (5.2) implies that the spectrum of $Q_{\theta,\Phi}^{\mathcal{O}}$ in this region is identical to the spectrum of $Q_\theta^{\Phi(\mathcal{O})}$. Moreover, $Q_{\theta,\Phi}^{\mathcal{O}}$ can be viewed as a restriction of $Q_{\theta,\Phi} := \Phi^* Q_\theta (\Phi^*)^{-1}$ to $\Gamma_\theta \setminus \mathcal{O}$ with Dirichlet boundary condition. A direct calculation shows that

$$A_\Phi := \Phi^* Q_\theta (\Phi^*)^{-1} - Q_\theta = \Phi^* Q (\Phi^*)^{-1} - Q = \sum_{|\alpha| \le 2} a_\alpha(x) \partial_x^\alpha, \tag{5.3}$$

where the coefficients a_α are supported in $B(0,R_1) \setminus \overline{B(0,R_0)} \subset \Gamma_\theta$. We note that $\|a_\alpha\|_\infty \le C \|\Phi - \mathrm{id}\,\|_{C^2}$, thus $A_\Phi = \mathcal{O}(\|\Phi - \mathrm{id}\,\|_{C^2}) : H^2(\Gamma_\theta) \to L^2(\Gamma_\theta)$.

Now we show that $\mathrm{Spec}(Q_\theta^{\mathcal{O}})$ can be moved by deforming the obstacle:

Lemma 5.1. *Suppose that the obstacle $\mathcal{O} \subset B(0,R_1)$ contains $\overline{B(0,R_0)}$ and that $-2\theta < \arg z_0 < 3\pi/2 + 2\theta$, then for any $\delta > 0$ there exists $\Phi \in \mathrm{Diff}(\mathcal{O})$ with $\|\Phi - \mathrm{id}\,\|_{C^2} < \delta$ such that $z_0 \notin \mathrm{Spec}(Q_\theta^{\Phi(\mathcal{O})})$.*

Proof. We may assume that $z_0 \in \mathrm{Spec}(Q_\theta^{\mathcal{O}})$, otherwise we can take $\Phi = \mathrm{id}$. Suppose that $Q_\theta^{\mathcal{O}}$ has exactly one eigenvalue in $D(z_0, 2r)$. For $D := D(z_0, r)$ we define

$$\Pi_{\mathcal{O}}(D) := -\frac{1}{2\pi i} \int_{\partial D} (Q_\theta^{\mathcal{O}} - \zeta)^{-1} d\zeta, \quad m_{\mathcal{O}}(D) := \mathrm{rank}\, \Pi_{\mathcal{O}}(D), \tag{5.4}$$

then $m_{\mathcal{O}}(D) = m_{\mathcal{O}}(z_0)$, where $m_{\mathcal{O}}(z_0)$ denotes the multiplicity of $z_0 \in \mathrm{Spec}(Q_\theta^{\mathcal{O}})$.

For $\delta > 0$ small, we put

$$\mathcal{U}_\delta(\mathcal{O}) := \{\Phi \in \mathrm{Diff}(\mathcal{O}) : \|\Phi - \mathrm{id}\,\|_{C^2(\mathbb{R}^n \setminus \mathcal{O})} < \delta\}.$$

It follows from (5.3) that $Q_{\theta,\Phi}^{\mathcal{O}} - Q_\theta^{\mathcal{O}} = O(\|\Phi - \mathrm{id}\,\|_{C^2}) : H^2(\Gamma_\theta \setminus \mathcal{O}) \to L^2(\Gamma_\theta \setminus \mathcal{O})$, thus for $\Phi \in \mathcal{U}_\delta(\mathcal{O})$ with δ sufficiently small,

$$(Q_{\theta,\Phi}^{\mathcal{O}} - \zeta)^{-1} = (Q_\theta^{\mathcal{O}} - \zeta)^{-1}(I + (Q_{\theta,\Phi}^{\mathcal{O}} - Q_\theta^{\mathcal{O}})(Q_\theta^{\mathcal{O}} - \zeta)^{-1})^{-1}, \quad \zeta \in \partial D,$$

exists and $\sup_{\zeta\in\partial D}\|(Q^{\mathcal{O}}_{\theta,\Phi}-\zeta)^{-1}-(Q^{\mathcal{O}}_{\theta}-\zeta)^{-1}\|_{L^2(\Gamma_\theta\setminus\mathcal{O})\to L^2(\Gamma_\theta\setminus\mathcal{O})}<C(\Omega)\delta$. We define

$$\Pi_\Phi(D):=-\frac{1}{2\pi i}\int_{\partial D}(Q^{\mathcal{O}}_{\theta,\Phi}-\zeta)^{-1}d\zeta,\quad m_\Phi(D):=\operatorname{rank}\Pi_\Phi(D),\tag{5.5}$$

then $\Pi_\Phi(D)$ and $\Pi_{\mathcal{O}}(D)$ have the same rank for any $\Phi\in\mathcal{U}_\delta(\mathcal{O})$ if δ is sufficiently small. Since $m_\Phi(D)=m_{\Phi(\mathcal{O})}(D)$ by (5.2), we conclude that

$$m_{\Phi(\mathcal{O})}(D)\text{ is constant for }\Phi\in\mathcal{U}_\delta(\mathcal{O})\text{ if }\delta\text{ is sufficiently small.}\tag{5.6}$$

We note that for every $\mathcal{O}$ and z_0, one of the following cases has to occur:

$$\forall\,\delta>0,\quad\exists\,\Phi\in\mathcal{U}_\delta(\mathcal{O})\ \text{ such that } m_{\Phi(\mathcal{O})}(z_0)<m_{\Phi(\mathcal{O})}(D),\tag{5.7}$$

or

$$\exists\,\delta>0,\ \text{such that}\quad\forall\,\Phi\in\mathcal{U}_\delta(\mathcal{O}),\ m_{\Phi(\mathcal{O})}(z_0)=m_{\Phi(\mathcal{O})}(D).\tag{5.8}$$

The first possibility means that by deforming $\mathcal{O}$ under an arbitrarily small Φ, we can obtain at least one eigenvalue of $Q^{\Phi(\mathcal{O})}_\theta$ other than z_0. The second possibility means that under any small deformation Φ, z_0 is the only eigenvalue of $Q^{\Phi(\mathcal{O})}_\theta$ in D and the maximal multiplicity persists.

Assuming (5.7) we can prove the lemma by induction on $m_{\mathcal{O}}(z_0)$. If $m_{\mathcal{O}}(z_0)=1$, (5.6) shows that $m_{\Phi(\mathcal{O})}(D)=1$ for $\Phi\in\mathcal{U}_\delta(\mathcal{O})$ with δ small. It then follows from (5.7) that we can find $\Phi\in\mathcal{U}_\delta(\mathcal{O})$ such that $m_{\Phi(\mathcal{O})}(z_0)<1$, i.e. $z_0\notin\operatorname{Spec}(Q^{\Phi(\mathcal{O})}_\theta)$. Assuming that we proved the lemma in the case $m_{\mathcal{O}}(z_0)<M$, we now assume that $m_{\mathcal{O}}(z_0)=M$. We note that for any $\Phi_1\in\operatorname{Diff}(\mathcal{O})$ and $\Phi_2\in\operatorname{Diff}(\Phi_1(\mathcal{O}))$,

$$\|\Phi_2\circ\Phi_1-\operatorname{id}\|_{C^2}\le C(\|\Phi_1-\operatorname{id}\|_{C^2}+\|\Phi_2-\operatorname{id}\|_{C^2}),$$

where C is a constant depending only on the dimension n. For any $\delta>0$, (5.7) implies that we can find $\Phi_1\in\operatorname{Diff}(\mathcal{O})$ with $\|\Phi_1-\operatorname{id}\|_{C^2}<\delta/2C$ such that $m_{\Phi_1(\mathcal{O})}(z_0)<M$. It then follows from our induction hypothesis that there exists $\Phi_2\in\operatorname{Diff}(\Phi_1(\mathcal{O}))$ with $\|\Phi_2-\operatorname{id}\|_{C^2}<\delta/2C$ such that $z_0\notin\operatorname{Spec}(Q^{\Phi_2(\Phi_1(\mathcal{O}))}_\theta)$. We now take $\Phi=\Phi_2\circ\Phi_1$, then $\Phi\in\mathcal{U}_\delta(\mathcal{O})$ and $z_0\notin\operatorname{Spec}(Q^{(\Phi(\mathcal{O})}_\theta)$.

It remains to show that (5.8) is impossible. For that, we shall argue by contradiction, assume that $m_{\mathcal{O}}(D)=M$ and that (5.8) holds. For $\Phi\in\mathcal{U}_\delta(\mathcal{O})$, we define

$$k(\Phi):=\min\{k:(Q^{\mathcal{O}}_{\theta,\Phi}-z_0)^k\Pi_\Phi(D)=0\},$$

then $1\le k(\Phi)\le M$. It follows from (5.2) and (5.5) that if $\|\Phi_j-\Phi\|_{C^{2M}}\to0$ and $(Q^{\mathcal{O}}_{\theta,\Phi_j}-z_0)^k\Pi_{\Phi_j}(D)=0$, then $(Q^{\mathcal{O}}_{\theta,\Phi}-z_0)^k\Pi_\Phi(D)=0$. We now put

$$k_0:=\max\{k(\Phi):\Phi\in\mathcal{U}_{\delta/2}(\mathcal{O})\},$$

and assume that the maximum is attained at $\Phi_0\in\mathcal{U}_{\delta/2}(\mathcal{O})$ i.e. $k(\Phi_0)=k_0$, then there exists $\delta'>0$ such that $\|\Phi-\Phi_0\|_{C^{2M}}<\delta'\Rightarrow k(\Phi)=k_0$. Henceforth, we can replace our original obstacle $\mathcal{O}$ with $\Phi_0(\mathcal{O})$, decrease δ and then assume by (5.8) that

$$\begin{gathered}(Q^{\mathcal{O}}_{\theta,\Phi}-z_0)^{k_0}\Pi_\Phi(D)=0,\quad (Q^{\mathcal{O}}_{\theta,\Phi}-z_0)^{k_0-1}\Pi_\Phi(D)\neq 0,\\ m_\Phi(z_0)=\operatorname{rank}\Pi_\Phi(D)=M,\quad \forall\,\Phi\in\operatorname{Diff}(\mathcal{O}),\ \|\Phi-\operatorname{id}\|_{C^{2M}}<\delta.\end{gathered}\tag{5.9}$$

Before proving that (5.9) is impossible we introduce a family of deformations in $\operatorname{Diff}(\mathcal{O})$ acting near a fixed point on $\partial\mathcal{O}$. For any fixed $x_0\in\partial\mathcal{O}$ and some $h_0>0$ small we can choose a family of functions $\chi_h\in\mathcal{C}^\infty(\partial\mathcal{O};[0,\infty))$ depending continuously in $h\in(0,h_0]$ with

$$\int_{\partial\mathcal{O}}\chi_h(x)dS(x)=1,\quad \operatorname{supp}\chi_h\subset B_{\partial\mathcal{O}}(x_0,h),\quad \forall\,h\in(0,h_0],\tag{5.10}$$

where $B_{\partial\mathcal{O}}(x_0,h)$ denotes the geodesic ball on $\partial\mathcal{O}$ with center x_0 and radius h. For each $h\in(0,h_0]$, we construct a smooth vector field $V_h\in\mathcal{C}^\infty_{\rm c}(\mathbb{R}^n;\mathbb{R}^n)$ with some small constant $\delta_h=\mathcal{O}(h^{2M+n-1})$ such that

$$\begin{gathered}V_h(x)=\delta_h\chi_h(x)\nu_g(x),\ \forall x\in\partial\mathcal{O},\quad \|V_h\|_{C^{2M}}<\varepsilon/2,\\ \operatorname{supp}V_h\subset B_{\mathbb{R}^n}(x_0,Ch)\text{ for some }C>0,\end{gathered}\tag{5.11}$$

where $\nu(x)$ is the normal vector at $x\in\partial\mathcal{O}$ pointing inward. Let $\varphi_h^t:\mathbb{R}^n\to\mathbb{R}^n$ be the flow generated by the vector field V_h. It follows from (5.11) that for every $h\in(0,h_0]$ there exists $t_0>0$ such that

$$\varphi_h^t\in\operatorname{Diff}(\mathcal{O}),\quad \|\varphi_h^t-\operatorname{id}\|_{C^{2M}}<\delta,\quad \forall\,t\in(-t_0,t_0).$$

Assuming (5.9) we can find $w\in L^2(\Gamma_\theta\setminus\mathcal{O})$ so that $u:=(Q^{\mathcal{O}}_\theta-z_0)^{k_0-1}\Pi_{\mathcal{O}}(D)w\neq 0$. For any fixed $x_0\in\partial\mathcal{O}$ and $h\in(0,h_0]$, we take $\Phi_t:=\varphi_h^t$, $t\in(-t_0,t_0)$ and put

$$u(t):=(\Phi_t{}^{-1})^*v(t),\quad v(t):=(Q^{\mathcal{O}}_{\theta,\Phi_t}-z_0)^{k_0-1}\Pi_{\Phi_t}(D)w.$$

In view of (5.2), $(Q^{\mathcal{O}}_{\theta,\Phi_t}-z_0)v(t)=0$ implies that

$$(Q_\theta-z_0)u(t)=0\quad\text{in }\Gamma_\theta\setminus\Phi_t(\mathcal{O}).\tag{5.12}$$

Since $\Phi_t(\mathcal{O})\subset\mathcal{O}$ for $t\geq 0$, we can restrict (5.12) to the region $\Gamma_\theta\setminus\mathcal{O}$ then differentiate it in t, by taking $t=0$, we obtain that

$$(Q_\theta-z_0)u'(0)=0\quad\text{in }\Gamma_\theta\setminus\mathcal{O}.\tag{5.13}$$

Recalling that $u(t,x)=v(t,\varphi_h^{-t}x)$ and $u(0)=v(0)=u$, we conclude from the flow equation that $u'(0)=v'(0)-\partial_x u\cdot V_h$, thus by (5.11) we have

$$u'(0)=-\delta_h\chi_h(x)\partial_{\nu_g}u,\quad\text{on }\partial\mathcal{O}.\tag{5.14}$$

We now multiply (5.13) by u then integrate it on $\Gamma_\theta \setminus \mathcal{O}$, then

$$\begin{aligned} 0 &= \int_{\Gamma_\theta\setminus\mathcal{O}} u\,(Q_\theta - z_0)u'(0) \\ &= \int_{\Gamma_\theta\setminus\mathcal{O}} u'(0)\,(Q_\theta - z_0)u + \int_{\Gamma_\theta\setminus\mathcal{O}} \sum_{j,k} \partial_j(u'(0)g^{jk}\partial_k u - ug^{jk}\partial_k u'(0)) \\ &= \int_{\partial\mathcal{O}} (u'(0)\,\partial_{\nu_g} u - u\,\partial_{\nu_g} u'(0))\,dS. \end{aligned} \tag{5.15}$$

It then follows from $u|_{\partial\mathcal{O}} = 0$ and (5.14) that

$$0 = \int_{\partial\mathcal{O}} \chi_h(x)(\partial_{\nu_g} u(x))^2 dS(x),$$

sending $h \to 0+$, we conclude from (5.10) that $\partial_{\nu_g} u(x_0) = 0$. We note that $x_0 \in \partial\mathcal{O}$ can be chosen arbitrarily, thus $\partial_{\nu_g} u|_{\partial\mathcal{O}} \equiv 0$. Putting $\tilde{u} := 1_{\mathcal{O}} \cdot 0 + 1_{\Gamma_\theta\setminus\mathcal{O}} \cdot u$, the same arguments as in the proof of Lemma 4.1 show that $\tilde{u} \in H^2(\Gamma_\theta)$ and $(Q_\theta - z_0)\tilde{u} = 0$ on Γ_θ. But unique continuation results for second order elliptic differential equations show that $\tilde{u} \equiv 0$, thus a contradiction. □

Now we consider the behavior of $\mathrm{Spec}(P^{\mathcal{O}})$ under the deformations of $\mathcal{O}$. In the notation of §2.2, for $\Phi \in \mathrm{Diff}(\mathcal{O})$, the pullback Φ^* gives an isomorphism between $\mathcal{H}^{\Phi(\mathcal{O})}$ and $\mathcal{H}^{\mathcal{O}}$, which also restricts to an isomorphism between $\mathcal{D}^{\Phi(\mathcal{O})}$ and $\mathcal{D}^{\mathcal{O}}$. Like (5.2) we define the deformed operator of $P^{\mathcal{O}}$ associate with Φ:

$$P^{\mathcal{O}}_\Phi := \Phi^* P^{\Phi(\mathcal{O})}(\Phi^*)^{-1}, \quad \text{with domain } \mathcal{D}^{\mathcal{O}}. \tag{5.16}$$

Since $(P^{\Phi(\mathcal{O})} + i)^{-1}$ is compact by Lemma 2.4, the same holds for $P^{\mathcal{O}}_\Phi$, it follows that $P^{\mathcal{O}}_\Phi$ has a discrete spectrum. Moreover, $\mathrm{Spec}(P^{\mathcal{O}}_\Phi)$ must be identical to $\mathrm{Spec}(P^{\Phi(\mathcal{O})})$, which lies in $\mathbb{R}$ due to the self-adjointness of $P^{\Phi(\mathcal{O})}$.

Before stating the deformation results for $\mathrm{Spec}(P^{\mathcal{O}})$, we notice that unlike Lemma 5.1, there is a subset of $\mathrm{Spec}(P^{\mathcal{O}})$ which is invariant under the deformations of the obstacle, that is the compactly supported embedded eigenvalues of P,

$$\mathrm{Spec}_{\mathrm{comp}}(P) := \{\lambda \in \mathbb{C} : \exists\, 0 \neq u \in \mathcal{D}_{\mathrm{comp}} \text{ such that } (P - \lambda)u = 0\}, \tag{5.17}$$

where $\mathcal{D}_{\mathrm{comp}} := \{u \in \mathcal{D} : u|_{\mathbb{R}^n\setminus B(0,R_0)} \in H^2_{\mathrm{comp}}(\mathbb{R}^n \setminus B(0,R_0))\}$. In view of the unique continuation results for second order elliptic differential equations, u in (5.17) must vanish on $\mathbb{R}^n \setminus B(0,R_0)$, thus $u \in \mathcal{D}^{\mathcal{O}}$ for any $\mathcal{O}$ containing $\overline{B(0,R_0)}$, which implies that $\mathrm{Spec}_{\mathrm{comp}}(P) \subset \mathrm{Spec}(P^{\mathcal{O}})$. The next lemma shows that any eigenvalue of $P^{\mathcal{O}}$ other than those compactly supported embedded eigenvalues of P can still be moved by deforming the obstacle:

Lemma 5.2. *Suppose that the obstacle $\mathcal{O} \subset B(0,R_1)$ contains $\overline{B(0,R_0)}$ and $z_0 \in \mathrm{Spec}(P^{\mathcal{O}})\setminus\mathrm{Spec}_{\mathrm{comp}}(P)$, then for any $\delta > 0$ there exists $\Phi \in \mathrm{Diff}(\mathcal{O})$ with $\|\Phi - \mathrm{id}\,\|_{C^2} < \delta$ such that $z_0 \notin \mathrm{Spec}(P^{\Phi(\mathcal{O})})$.*

Proof. The proof is similar to Lemma 5.1 except that we need a different approach from (5.15) since the integration by parts is not available in the black box. Suppose that $z_0 \in \operatorname{Spec}(P^{\mathcal{O}})$ with multiplicity $m^P_{\mathcal{O}}(z_0)$ and that $P^{\mathcal{O}}$ has exactly one eigenvalue in $D(z_0, 2r)$. For $D := D(z_0, r)$ we put

$$\Pi^P_{\mathcal{O}}(D) := -\frac{1}{2\pi i}\int_{\partial D}(P^{\mathcal{O}} - \zeta)^{-1}d\zeta, \quad m^P_{\mathcal{O}}(D) := \operatorname{rank}\Pi^P_{\mathcal{O}}(D).$$

Using (2.12) and (5.3) we can deduce that $\partial D \ni \zeta \mapsto (P^{\mathcal{O}}_{\Phi} - \zeta)^{-1}$ exists for $\Phi \in \mathcal{U}_\delta(\mathcal{O})$ with δ small enough, then we define

$$\Pi^P_{\Phi}(D) := -\frac{1}{2\pi i}\int_{\partial D}(P^{\mathcal{O}}_{\Phi} - \zeta)^{-1}d\zeta, \quad m^P_{\Phi}(D) := \operatorname{rank}\Pi^P_{\Phi}(D) = m^P_{\Phi(\mathcal{O})}(D).$$

We remark that $m^P_{\mathcal{O}}(D)$ is also invariant under small deformations of obstacles:

$$m^P_{\Phi(\mathcal{O})}(D) \text{ is constant for } \Phi \in \mathcal{U}_\delta(\mathcal{O}) \text{ if } \delta \text{ is sufficiently small.} \tag{5.18}$$

In view of the proof of Lemma 5.1, it is enough to exclude the following case:

$$\exists\, \delta > 0, \text{ such that } \quad \forall\, \Phi \in \mathcal{U}_\delta(\mathcal{O}),\ m^P_{\Phi(\mathcal{O})}(z_0) = m^P_{\Phi(\mathcal{O})}(D). \tag{5.19}$$

Again we argue by contradiction, assume that (5.19) holds and $m^P_{\mathcal{O}}(D) = M \geq 1$. We remark that unlike the proof of Lemma 5.1, the self-adjointness of $P^{\Phi(\mathcal{O})}$ implies that $(P^{\Phi(\mathcal{O})} - z_0)\Pi^P_{\Phi(\mathcal{O})}(D) = 0$ thus $(P^{\mathcal{O}}_{\Phi} - z_0)\Pi^P_{\Phi}(D) = 0$ for any $\Phi \in \mathcal{U}_\delta(\mathcal{O})$. We now choose $w \in \mathcal{H}^{\mathcal{O}}$ such that $u := \Pi^P_{\mathcal{O}}(D)w \neq 0$. For any fixed $x_0 \in \partial\mathcal{O}$ and $h \in (0, h_0]$, we set $\Phi_t := \varphi^t_h$ where φ^t_h is the flow generated by V_h given in (5.11), there exists $t_0 > 0$ such that $\Phi_t \in \mathcal{U}_\delta(\mathcal{O})$ for all $-t_0 < t < t_0$. Let

$$v(t) := \Pi^P_{\Phi_t}(D)w \in \mathcal{D}^{\mathcal{O}}, \quad u(t) := (\Phi_t^{-1})^* v(t),$$

we have $(P^{\mathcal{O}}_{\Phi_t} - z_0)v(t) = 0$, thus $(P^{\Phi_t(\mathcal{O})} - z_0)u(t) = 0$. Recalling (2.12) we obtain that for some $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{O})$, $\psi = 1$ near $\overline{B(0, R_0)}$ and $t_0 > 0$ small enough,

$$\forall\, t \in (-t_0, t_0), \quad P(\psi u(t)) + Q((1-\psi)u(t)) - z_0 u(t) = 0 \quad \text{in } \Phi_t(\mathcal{O}). \tag{5.20}$$

Since $\Phi_t(\mathcal{O}) \supset \mathcal{O}$ for $t \leq 0$, we can restrict (5.20) to $\mathcal{O}$ and differentiate it in t, by taking $t = 0$, we have

$$P(\psi u'(0)) + Q((1-\psi)u'(0)) - z_0 u'(0) = 0 \quad \text{in } \mathcal{O}. \tag{5.21}$$

Next we compute the inner product of the left hand side and u on the Hilbert space $\mathcal{H}^{\mathcal{O}}$ defined by (2.10). For that, choose $\psi_j \in \mathcal{C}^\infty_{\mathrm{c}}(\mathcal{O})$, $\psi_j = 1$ near $\overline{B(0, R_0)}$, so that

$$\psi_1 = 1 \text{ near } \operatorname{supp}\psi, \quad \psi = 1 \text{ near } \operatorname{supp}\psi_2. \tag{5.22}$$

Then we have, using the self-adjointness of P,

$$\langle P(\psi u'(0)), u\rangle_{\mathcal{H}^{\mathcal{O}}} = \langle P(\psi u'(0)), \psi_1 u\rangle_{\mathcal{H}} = \langle \psi u'(0), P(\psi_1 u)\rangle_{\mathcal{H}},$$

and $\langle Q((1-\psi)u'(0)), u\rangle_{\mathcal{H}^{\mathcal{O}}} = \langle Q((1-\psi)u'(0)), (1-\psi_2)u\rangle_{L^2(\mathcal{O})}$. Recalling (5.14), integration by parts as in (5.15) shows that

$$
\begin{aligned}
&\langle Q((1-\psi)u'(0)), (1-\psi_2)u\rangle_{L^2(\mathcal{O})} - \langle (1-\psi)u'(0), Q((1-\psi_2)u)\rangle_{L^2(\mathcal{O})} \\
&= \int_{\mathcal{O}} \sum_{j,k} \partial_j((1-\psi)u'(0)g^{jk}\partial_k((1-\psi_2)\bar{u}) - (1-\psi_2)\bar{u}g^{jk}\partial_k((1-\psi)u'(0))) \\
&= \int_{\partial\mathcal{O}} -u'(0)\partial_{\nu_g}\bar{u} + \bar{u}\partial_{\nu_g}u'(0) = \int_{\partial\mathcal{O}} \delta_h\chi_h|\partial_{\nu_g}u|^2.
\end{aligned}
$$

It follows from (2.12) and (5.22) that

$$
\langle \psi u'(0), P(\psi_1 u)\rangle_{\mathcal{H}} = \langle u'(0), \psi(P^{\mathcal{O}}u - Q((1-\psi_1)u))\rangle_{\mathcal{H}^{\mathcal{O}}} = \langle u'(0), \psi P^{\mathcal{O}}u\rangle_{\mathcal{H}^{\mathcal{O}}};
$$

and that

$$
\begin{aligned}
\langle (1-\psi)u'(0), Q((1-\psi_2)u)\rangle_{L^2(\mathcal{O})} &= \langle u'(0), (1-\psi)(P^{\mathcal{O}}u - P(\psi_2 u))\rangle_{\mathcal{H}^{\mathcal{O}}} \\
&= \langle u'(0), (1-\psi)P^{\mathcal{O}}u\rangle_{\mathcal{H}^{\mathcal{O}}}.
\end{aligned}
$$

We now conclude from (5.21) and all the calculation above that

$$
0 = \langle u'(0), (P^{\mathcal{O}} - z_0)u\rangle_{\mathcal{H}^{\mathcal{O}}} + \int_{\partial\mathcal{O}} \delta_h\chi_h|\partial_{\nu_g}u|^2 = \int_{\partial\mathcal{O}} \delta_h\chi_h|\partial_{\nu_g}u|^2.
$$

It follows that $\partial_{\nu_g}u(x_0) = 0$. Since $x_0 \in \partial\mathcal{O}$ can be chosen arbitrarily, we obtain that $\partial_{\nu_g}u|_{\partial\mathcal{O}} \equiv 0$. Putting $\tilde{u} := 1_{\mathcal{O}}u + 1_{\mathbb{R}^n\setminus\mathcal{O}}\cdot 0$, the same arguments as in the proof of Lemma 4.1 show that $\tilde{u} \in \mathcal{D}$ and $(P - z_0)\tilde{u} = 0$, which would imply that $z_0 \in \mathrm{Spec}_{\mathrm{comp}}(P)$, a contradiction. □

6. Proof of convergence

Before proving the convergence of eigenvalues of P_ε to resonances as $\varepsilon \to 0+$, we recall a basic estimate of decay of the Green function of $Q_\theta^{\mathcal{O}}$ off the diagonal $\{(x,x) : x \in \Gamma_\theta \setminus \mathcal{O}\}$. For a detailed account see Shubin [Sh92] and references given there.

Lemma 6.1. *Suppose that the obstacle $\mathcal{O} \subset B(0,R_1)$ contains $\overline{B(0,R_0)}$ and that $z_0 \notin \mathrm{Spec}(Q_\theta^{\mathcal{O}})$ with $-2\theta < \arg z_0 < 3\pi/2 + 2\theta$. The Schwartz kernel of the resolvent $(Q_\theta^{\mathcal{O}} - z_0)^{-1} : L^2(\Gamma_\theta \setminus \mathcal{O}) \to L^2(\Gamma_\theta \setminus \mathcal{O})$ is denoted by $G(z_0; x_\theta, y_\theta)$, where $x_\theta = f_\theta(x)$ is the parametrization on Γ_θ. Then there exists $\beta > 0$ such that for every $\delta > 0$ there exists $C_\delta > 0$ such that*

$$
|G(z_0; f_\theta(x), f_\theta(y))| \le C_\delta\, e^{-\beta|x-y|} \quad \textit{if} \quad |x-y| > \delta.
$$

Proof. Identifying Γ_θ and $\mathbb{R}^n$ by means of f_θ, the pullback f_θ^* gives an isomorphism between $L^2(\Gamma_\theta \setminus \mathcal{O})$ and $L^2(\mathbb{R}^n \setminus \mathcal{O})$ since there exists $C > 0$ such that

$$
C^{-1} < |\det df_\theta(x)| = |x|^{1-n}|g_\theta(|x|)|^{n-1}|g_\theta'(|x|)| < C, \quad \text{for all } x.
$$

Let $\tilde{Q}_\theta^{\mathcal{O}} := f_\theta^* Q_\theta^{\mathcal{O}} (f_\theta^*)^{-1} : L^2(\mathbb{R}^n \setminus \mathcal{O}) \to L^2(\mathbb{R}^n \setminus \mathcal{O})$ then $\tilde{Q}_\theta^{\mathcal{O}}$ is elliptic and equipped with the domain $H^2(\mathbb{R}^n \setminus \mathcal{O}) \cap H_0^1(\mathbb{R}^n \setminus \mathcal{O})$. Moreover, $(\tilde{Q}_\theta^{\mathcal{O}} - z_0)^{-1}$ exists and we denote its Schwartz kernel by $\tilde{G}(z_0; x, y)$, $x, y \in \mathbb{R}^n \setminus \mathcal{O}$, i.e. $\tilde{G}(z_0; x, y) = [(\tilde{Q}_\theta^{\mathcal{O}} - z_0)^{-1} \delta_y(\cdot)](x)$ where δ_y is the Dirac function supported at y.

The same arguments as in [Sh92, Appendix 1] show that there exists $\beta > 0$ such that for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$|\tilde{G}(z_0; x, y)| \leq C_\delta \, e^{-\beta|x-y|} \quad \text{if} \quad |x - y| > \delta.$$

We remark that the assumption in [Sh92, Appendix 1.1] that the manifold M is complete can be dropped if we introduce $\tilde{d}(x, y)$, the substitute with smoothness properties for the distance $|x - y|$, on the whole $\mathbb{R}^n$ then restrict it to $\mathbb{R}^n \setminus \mathcal{O}$. The remaining arguments in [Sh92, Appendix 1.2] are still valid if we replace M by $\mathbb{R}^n \setminus \mathcal{O}$.

Using $(\tilde{Q}_\theta^{\mathcal{O}} - z_0)^{-1} = f_\theta^* (Q_\theta^{\mathcal{O}} - z_0)^{-1} (f_\theta^*)^{-1}$ we obtain that

$$G(z_0; f_\theta(x), f_\theta(y)) = (\det df_\theta(y))^{-1} \tilde{G}(z_0; x, y), \quad x, y \in \mathbb{R}^n \setminus \mathcal{O},$$

the desired estimate of $G(z_0; x_\theta, y_\theta)$ then follows from the estimate of $\tilde{G}(z_0; x, y)$. $\square$

We now state a more precise version of Theorem 1:

Theorem 2. *Suppose that $\Omega \Subset \{z : -2\theta_0 < \arg z < 3\pi/2 + 2\theta_0\}$. Then there exists $\delta_0 = \delta_0(\Omega) > 0$ such that $\forall\, 0 < \delta < \delta_0$, $\exists\, \varepsilon_\delta > 0$ such that*

$$0 < \varepsilon < \varepsilon_\delta \implies \operatorname{Spec}(P_\varepsilon) \cap \Omega_\delta \subset \bigcup_{j=1}^{J} D(z_j, \delta), \tag{6.1}$$

where $\Omega_\delta := \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > \delta\}$ and $z_1, \cdots, z_J$ are the resonances of P in Ω. Furthermore, for each resonance z_j with the multiplicity $m(z_j)$ given by (2.9),

$$\# \operatorname{Spec}(P_\varepsilon) \cap D(z_j, \delta) = m(z_j), \quad \forall\, 0 < \varepsilon < \varepsilon_\delta, \tag{6.2}$$

where the eigenvalue in $\operatorname{Spec}(P_\varepsilon)$ is counted with multiplicity defined in (3.9).

Proof. First we put $\delta_0 = \frac{1}{2} \min_{1 \leq j \leq J} \operatorname{dist}(z_j, \partial\Omega)$ and fix $\theta \in [0, \theta_0)$ such that $\Omega \Subset \{z : -2\theta < \arg z < 3\pi/2 + 2\theta\}$. To prove (6.1) we argue by contradiction. Suppose that there exist some $\delta < \delta_0$ and a sequence $\varepsilon_k \to 0+$ such that

$$\exists\, z_k \in \operatorname{Spec}(P_{\varepsilon_k}) \cap \Omega_\delta \setminus \bigcup_{j=1}^{J} D(z_j, \delta), \quad k = 1, 2, \cdots$$

Then there exists a subsequence $z_{n_k} \to z_0$, as $k \to \infty$, for some $z_0 \in \overline{\Omega_\delta} \setminus \bigcup_{j=1}^{J} D(z_j, \delta)$. Since $z_0 \in \Omega$, we see that z_0 is not a resonance, thus $\mathcal{P}_\theta - z_0$ is invertible by definition. We may assume that $D(z_0, r)$ is disjoint with $\operatorname{Spec}(\mathcal{P}_\theta)$ for some $r > 0$, it then follows from Lemma 3.5 that $\operatorname{Spec}(\mathcal{P}_{\varepsilon,\theta}) \cap D(z_0, r) = \emptyset$ for ε small enough. However, Lemma

3.4 shows that $\operatorname{Spec}(P_{\varepsilon_{n_k},\theta}) = \operatorname{Spec}(P_{\varepsilon_{n_k}}) \ni z_{n_k} \to z_0$ while $\varepsilon_{n_k} \to 0+$, which gives a contradiction.

It remains to prove (6.2). For each resonance z_j, let

$$V_j := \{u \in \mathcal{D}_{\mathrm{comp}} : (P - z_j)u = 0\},$$

then V_j is finite dimensional and $V_j \neq \{0\}$ if and only if $z_j \in \operatorname{Spec}_{\mathrm{comp}}(P)$. We remark that V_j is a subspace of $\mathcal{H}_{R_0}$ given in (1.1), as a consequence of the unique continuation results for second order elliptic equations. The self-adjointness of P implies that $V_1 \perp \cdots \perp V_J$ in the Hilbert space $\mathcal{H}$. Putting $V_0 := V_1 \oplus \cdots \oplus V_J$, $\mathcal{H}$ admits the following orthogonal decomposition:

$$\mathcal{H} = V_0 \oplus \tilde{\mathcal{H}}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)). \tag{6.3}$$

Let $\Pi_0 : \mathcal{H} \to V_0$ be the orthogonal projection. Since V_0 is an invariant subspace under P, we can introduce the restriction of P as follows:

$$\tilde{P} : \tilde{\mathcal{H}}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)) \to \tilde{\mathcal{H}}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)), \quad \tilde{P}u := (I - \Pi_0)Pu.$$

If we replace $\mathcal{H}_{R_0}$ with $\tilde{\mathcal{H}}_{R_0}$ and replace P by $\tilde{P}$, which is also self-adjoint with domain $\tilde{\mathcal{D}} := (I - \Pi_0)\mathcal{D}$, it is easy to see that the assumptions (1.2) – (1.5) are still satisfied. Recalling the definition of resonances introduced in §2.1, any resonance of $\tilde{P}$ must also be a resonance of P and we have

$$m(z_j) = \operatorname{rank} \oint_{z_j} (z - \tilde{P}_\theta)^{-1} dz + \dim V_j.$$

Note that $V_j \neq \{0\}$ implies that $z_j \in \operatorname{Spec}(P_\varepsilon)$ for every $\varepsilon > 0$. Putting $\tilde{P}_\varepsilon := \tilde{P} - i\varepsilon(1 - \chi(x))x^2$, it follows that

$$\# \operatorname{Spec}(P_\varepsilon) \cap D(z_j, \delta) = \# \operatorname{Spec}(\tilde{P}_\varepsilon) \cap D(z_j, \delta) + \dim V_j, \quad \forall\, \varepsilon > 0,$$

while both sides are counted with multiplicities. Hence it is enough to establish (6.2) for $\tilde{P}$. In other words, it suffices to prove (6.2) in the case that P has no compactly supported embedded eigenvalues in Ω.

Now we assume that $\operatorname{Spec}_{\mathrm{comp}}(P) \cap \Omega = \emptyset$. Lemma 5.1 and 5.2 show that there exists an obstacle $\mathcal{O} \subset B(0, R_1)$ containing $\overline{B(0, R_0)}$ such that χ in (1.8) is equal to 1 near $\mathcal{O}$ and that $z_j \notin \operatorname{Spec}(P^{\mathcal{O}}) \cup \operatorname{Spec}(Q^{\mathcal{O}}_\theta)$, $j = 1, \cdots, J$. Then we can decrease δ_0 such that $\operatorname{Spec}(P^{\mathcal{O}})$ and $\operatorname{Spec}(Q^{\mathcal{O}}_\theta)$ are disjoint with $\bigcup_{j=1}^J D(z_j, 2\delta_0)$. For each $\delta \in (0, \delta_0)$, we can also decrease ε_δ in (6.1) such that

$$\forall\, 0 \le \varepsilon < \varepsilon_\delta, \quad \bigcup_{j=1}^J D(z_j, 2\delta) \cap \operatorname{Spec}(Q^{\mathcal{O}}_{\varepsilon,\theta}) = \emptyset.$$

This follows from Lemma 3.5 applied with $\mathcal{P}_\theta = Q_\theta^{\mathcal{O}}$ and $\Omega = \bigcup_{j=1}^J D(z_j, 2\delta)$. Hence the Dirichlet-to-Neumann operators $\widehat{\mathcal{N}}_{\varepsilon,\theta}(z)$, $0 \le \varepsilon < \varepsilon_\delta$ introduced in §4, are well-defined for $z \in \bigcup_{j=1}^J D(z_j, 2\delta)$. In view of (6.1), Lemma 3.4 and 4.2 we obtain that $\partial D(z_j,\delta) \ni w \mapsto \widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1}$ exists and that for all $0 < \varepsilon < \varepsilon_\delta$, $j = 1, \cdots, J$,

$$\# \operatorname{Spec}(P_\varepsilon) \cap D(z_j,\delta) = \frac{1}{2\pi i} \operatorname{tr} \int_{\partial D(z_j,\delta)} \widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1} \partial_w \widehat{\mathcal{N}}_{\varepsilon,\theta}(w) dw. \tag{6.4}$$

In order to apply the Gohberg–Sigal–Rouché theorem, we need the estimate:

$$\forall\, 0 < \varepsilon < \varepsilon_\delta, \quad \|\widehat{\mathcal{N}}_{\varepsilon,\theta}(w) - \widehat{\mathcal{N}}_\theta(w)\|_{H^{3/2}(\partial\mathcal{O}) \to H^{3/2}(\partial\mathcal{O})} < 1, \quad w \in \partial D(z_j,\delta), \tag{6.5}$$

here we write $\widehat{\mathcal{N}}_\theta(\cdot) = \widehat{\mathcal{N}}_{0,\theta}(\cdot)$ for simplicity. To obtain this estimate, we first choose E^{out} in (4.6) such that $\chi = 1$ near $\operatorname{supp} E^{\mathrm{out}}\varphi$ for any $\varphi \in H^{3/2}(\partial\mathcal{O})$, then (4.6) reduces to $\mathcal{N}_{\varepsilon,\theta}^{\mathrm{out}}(z)\varphi = \partial_{\nu_g}(E^{\mathrm{out}}\varphi - (Q_{\varepsilon,\theta}^{\mathcal{O}} - z)^{-1}(Q - z)E^{\mathrm{out}}\varphi)$. Therefore,

$$(\widehat{\mathcal{N}}_{\varepsilon,\theta}(w) - \widehat{\mathcal{N}}_\theta(w))\varphi = \langle D_{\partial\mathcal{O}}\rangle^{-1} \partial_{\nu_g}((Q_\theta^{\mathcal{O}} - w)^{-1} - (Q_{\varepsilon,\theta}^{\mathcal{O}} - w)^{-1})(Q - w)E^{\mathrm{out}}\varphi.$$

Choosing $\psi \in \mathcal{C}_{\mathrm{c}}^\infty(\Gamma_\theta \setminus \mathcal{O})$ such that $\psi = 1$ near $\operatorname{supp} E^{\mathrm{out}}\varphi$, $\forall \varphi \in H^{3/2}(\partial\mathcal{O})$ and that $\chi = 1$ near $\operatorname{supp}\psi$, (6.5) then follows from the following estimate: for $w \in \partial D(z_j,\delta)$,

$$((Q_\theta^{\mathcal{O}} - w)^{-1} - (Q_{\varepsilon,\theta}^{\mathcal{O}} - w)^{-1})\psi = O_\delta(\varepsilon) : L^2(\Gamma_\theta \setminus \mathcal{O}) \to H^2(\Gamma_\theta \setminus \mathcal{O}). \tag{6.6}$$

To obtain (6.6), we denote the Schwartz kernel of the operator $(1-\chi)x_\theta^2(Q_{\varepsilon,\theta}^{\mathcal{O}} - w)^{-1}\psi$ by $K(w; x_\theta, y_\theta)$. In the notation of Lemma 6.1, we have

$$K(w; f_\theta(x), f_\theta(y)) = (1 - \chi(x)) f_\theta(x)^2 G(w; f_\theta(x), f_\theta(y))\psi(y).$$

It follows from Lemma 6.1 that there exists $\beta_\delta > 0$ such that for all $w \in \partial D(z_j,\delta)$, $j = 1, \cdots, J$, $|K(w; f_\theta(x), f_\theta(y))| \le C|x|^2 e^{-\beta_\delta|x-y|}\psi(y)$, thus

$$\sup_{x_\theta} \int_{\Gamma_\theta\setminus\mathcal{O}} |K(w; x_\theta, y_\theta)||dy_\theta| \le C_\delta, \quad \sup_{y_\theta} \int_{\Gamma_\theta\setminus\mathcal{O}} |K(w; x_\theta, y_\theta)||dx_\theta| \le C_\delta.$$

The Schur test shows that $(1-\chi)x_\theta^2(Q_{\varepsilon,\theta}^{\mathcal{O}} - w)^{-1}\psi = O_\delta(1) : L^2(\Gamma_\theta \setminus \mathcal{O}) \to L^2(\Gamma_\theta \setminus \mathcal{O})$. Hence we can write

$$((Q_\theta^{\mathcal{O}} - w)^{-1} - (Q_{\varepsilon,\theta}^{\mathcal{O}} - w)^{-1})\psi = -i\varepsilon(Q_{\varepsilon,\theta}^{\mathcal{O}} - w)^{-1}(1-\chi)x_\theta^2(Q_\theta^{\mathcal{O}} - w)^{-1}\psi.$$

It remains to show that for $\varepsilon_\delta > 0$ small enough,

$$(Q_{\varepsilon,\theta}^{\mathcal{O}} - w)^{-1} = O_\delta(1) : L^2(\Gamma_\theta \setminus \mathcal{O}) \to H^2(\Gamma_\theta \setminus \mathcal{O}), \quad w \in \bigcup_{j=1}^J \partial D(z_j,\delta),\ 0 < \varepsilon < \varepsilon_\delta.$$

This follows from Lemma 3.5 with $\mathcal{P}_\theta = Q_\theta^{\mathcal{O}}$ and $\Omega = \bigcup_{j=1}^J \partial D(z_j,\delta)$. Using (6.6) we can decrease ε_δ such that (6.5) holds for $j = 1, \cdots, J$. Now we apply the Gohberg–Sigal–Rouché theorem to conclude that for all $0 < \varepsilon < \varepsilon_\delta$ and $j = 1, \cdots, J$,

$$\frac{1}{2\pi i} \operatorname{tr} \int_{\partial D(z_j,\delta)} \widehat{\mathcal{N}}_{\varepsilon,\theta}(w)^{-1} \partial_w \widehat{\mathcal{N}}_{\varepsilon,\theta}(w) dw = \frac{1}{2\pi i} \operatorname{tr} \int_{\partial D(z_j,\delta)} \widehat{\mathcal{N}}_\theta(w)^{-1} \partial_w \widehat{\mathcal{N}}_\theta(w) dw.$$

Finally, using Lemma 4.2, (6.4) and the equation above, we obtain (6.2). □

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Email address: xiong@math.berkeley.edu

Department of Mathematics, University of California, Berkeley, CA 94720, USA