Worksheet 9 (Feb. 10)

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1 Review

DEFINITIONS

- Inverse of linear transformation, inverse of matrix;

- determinant of matrix.

METHODS AND IDEAS

- A transformation is **invertible** if and only if it is **bijective**, i.e. both one-to-one and onto.
- To compute the inverse of a matrix A, apply row reduction

$$\begin{pmatrix} A & I_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} I_n & A^{-1} \end{pmatrix}.$$

i.e., when A is reduced to I_n, what appears on RHS is the A^{-1}.

Theorem 1. [Equivalent conditions for invertibility]
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and A be its standard matrix, then all the following statements are equivalent conditions of invertibility of T:

(linear transformation) $\Leftrightarrow$ *T is invertible* $\Leftrightarrow$ *T is bijective*

(vector) $\Leftrightarrow$ *$T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ are linearly independent and span the entire $\mathbb{R}^n$*

(matrix) $\Leftrightarrow$ *A is invertible* $\Leftrightarrow$ *A has n pivots (and thus one in each row and each column)* $\Leftrightarrow$ $\det(A) \neq 0$

(linear system) $\Leftrightarrow$ *$A\mathbf{x} = \mathbf{b}$ has unique solution for any $\mathbf{b} \in \mathbb{R}^n$.*

2 Problems

Example 1. Compute the determinants below

$$\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix}, \qquad \begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \\ 0 & -2 & 4 \end{vmatrix}.$$

Example 2. Consider the 2×2 matrices

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

(a) Find $A^{-1}B$.
(b) Solve the linear systems $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$, for $\mathbf{b}_1$ and $\mathbf{b}_2$ the two column vectors of B.

Example 3. True or false.

( ) The product of two invertible matrices is still invertible.

( ) The composition of two invertible linear transformations is still invertible.

( ) The inverse of an invertible matrix is invertible.

( ) An upper-triangular matrix is invertible.

( ) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a surjective linear transformation and A be its standard matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ has unique solution for any $\mathbf{b} \in \mathbb{R}^n$.

( ) If $C = AB$ where A is 3×2 and B is 2×3, then C can never be invertible.