Worksheet 6 (Feb. 3)

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1 Review

DEFINITIONS

- Trichotomy of linear systems (vector language);
- linear transformation, domain, codomain;
- standard basis of $\mathbb{R}^n$, matrix of a linear transformation.

METHODS AND IDEAS

- A map $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation of vectors $\Leftrightarrow$ it is a matrix multiplication, i.e. we can find an $m \times n$ matrix such that $f(\mathbf{x}) = A\mathbf{x}$, for any $\mathbf{x} \in \mathbb{R}^n$. The columns of A are exactly $f(\mathbf{e}_1), f(\mathbf{e}_2), \cdots, f(\mathbf{e}_n)$.
- Thus in particular, the linear transformation f is completely determined by the **images** $f(\mathbf{e}_1), f(\mathbf{e}_2), \cdots, f(\mathbf{e}_n)$ of the standard basis of $\mathbb{R}^n$!!!

2 Problems

Example 1. Determine if the following maps are linear transformations.
(1) $f : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation of vectors counterclockwise by $\pi/3$ about the origin.

(2) $g : \mathbb{R}^2 \to \mathbb{R}^2$ sending any vector $\mathbf{x}$ to $\mathbf{x} + \mathbf{e}_1$.

(3) $h : \mathbb{R}^3 \to \mathbb{R}^2$ with

$$h \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ z \end{pmatrix}.$$

(4) $k : \mathbb{R} \to \mathbb{R}$ with

$$g(x) = |x|.$$

Example 2. Compare the two linear transformations

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad g : \mathbb{R}^2 \to \mathbb{R}^2$$

$$f \begin{pmatrix} x \\ y \end{pmatrix} = x, \qquad g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Why are they different linear transformations?

Example 3. Assume that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation. Consider the following vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}.$$

(1) Assume further that $T(\mathbf{e}_1) = \mathbf{u}$ and $T(\mathbf{e}_2) = \mathbf{v}$. Find the matrix of T.
(2) Assume instead that $T(\mathbf{a}) = \mathbf{u}$ and $T(\mathbf{b}) = \mathbf{v}$. Find the matrix of T. *Hint: why is T alsodetermined completely here?*