Worksheet 4 (Jan. 29)

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1 Review

DEFINITIONS

- matrix-vector product;

- 4 equivalent ways of writing a linear system, homogeneous system, inhomogeneous system;

- linear independence.

METHODS AND IDEAS

- Expanded **criterion for uniqueness**: provided that solutions of a **linear system** $A\mathbf{x} = \mathbf{b}$ exist, its solution is unique $\Leftrightarrow$ the homogenized system $A\mathbf{x} = \mathbf{0}$ has unique solution (the trivial solution $\mathbf{0}$) $\Leftrightarrow$ the column **vectors** of its coefficient matrix A are linear independent $\Leftrightarrow$ all columns of A have pivot positions (i.e. no free variable) in **REF**. [See the P12 of the professor's notes for a beautiful chart.]
- Solution sets of $A\mathbf{x} = \mathbf{b}$ and of $A\mathbf{x} = \mathbf{0}$ in $\mathbb{R}^m$ are "parallel planes", when both systems are consistent.

Remark 1. In general, if a homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then an inhomogeneous system $A\mathbf{x} = \mathbf{b}$ with the same coefficient matrix can have 0 or 1 solution.

Remark 2. By the relation between the coefficient matrix and its column vectors, for the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ in $\mathbb{R}^m$, they cannot be independent if $n > m$, and they cannot span the entire $\mathbb{R}^m$ if $n < m$. Moreover, if $m = n$, then:

$$\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \text{ are linearly independent } \Leftrightarrow \text{ span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} = \mathbb{R}^m.$$

2 Problems

Example 1. Consider vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ in $\mathbb{R}^3$

$$\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Are they linear independent? Can $\mathbf{u}_3$ be written as a linear combination of the other two?

Example 2. Discuss: about linear dependence. *Note that for determining linear independence it does not matter whether you are picturing these vectors in $\mathbb{R}^3$ or $\mathbb{R}^{100}$, as the extra "directions" are irrelevant here.*
(a) If the set of a single vector $\{\mathbf{v}\}$ is linearly dependent, what do we know of $\mathbf{v}$?

(b) If the set of two vectors $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent, what do we know of $\mathbf{u}, \mathbf{v}$?

(c) If the set of three vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, what do we know of $\mathbf{u}, \mathbf{v}, \mathbf{w}$?

Example 3. Find possible values of c such that the following three vectors are linearly dependent

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ c \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ c \end{bmatrix}.$$