Worksheet 31 (April 28)

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1 Review

DEFINITIONS

- inner product: bilinear, commutative, positive definite;
- piece-wise continuous function, even function, odd function;
- Fourier series, Fourier expansion of a function

METHODS AND IDEAS

Idea 1. *The space V_π of piece-wise continuous functions is an* ***inner product space****, under the inner product*

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)dx.$$

This is because $\langle \cdot, \cdot\rangle$ defined as above is indeed bilinear, commutative and positive definite. V_π has the following properties:

- *It is* ***infinite dimensional****.*
- *It has two distinguished subspaces V_π^{even} and V_π^{odd}. The two subspaces are* ***orthogonal****.*
- *V_π^{even} has an orthogonal basis $\{\cos nx\}_{n\geqslant 0}$, and V_π^{odd} has an orthogonal basis $\{\sin nx\}_{n>0}$. Altogether they form a basis of V_π.*

Theorem 1. *The Fourier expansion of $f(x)$ is given by*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$$

Remark 1. The two functions

$$f(x) \quad \text{and} \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

are not completely equal. In fact, if $f(x)$ is piece-wise continuous, they are equal "**almost everywhere**". The discrepancy exists because V_π is infinite-dimensional and the basis $\{1, \cos x, \sin x, \cdots\}$ is not "**complete**".

2 Problems

Example 1. Find a linear transformation $T : V_\pi \to V_\pi$ such that V_π^{even} and V_π^{odd} are both eigenspaces of T.

Example 2. True or false,

( ) If $f(x)$ is an odd piece-wise continuous function, then $a_n = 0$ for all $n \geqslant 0$ in its Fourier expansion.

( ) The Fourier expansion of a piece-wise continuous function is unique.

( ) Let A be a 2×2 symmetric matrix which has two distinct eigenvalues 2 and 3, then the pairing

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}$$

gives an inner product on $\mathbb{R}^2$.

Example 3. Consider the inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{P}^2$ defined by

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)dx.$$

(a) Find $\langle 1+x, x^2\rangle$.
(b) Let $W = \operatorname{sgn}\{1, x^2\}$ and $u(x) = 2+3x+4x^2$. Find $\operatorname{Proj}_W u(x)$ under the above inner product.

Example 4. Find the Fourier expansion of

$$f(x) = \operatorname{sgn}(\cos x)$$

on $[-\pi,\pi]$. Here sgn means taking the sign. In other words, $f(x) = 1$ when $\cos x > 0$, $f(x) = -1$ when $\cos x < 0$, and $f(x) = 0$ when $\cos x = 0$.

Example 5. *This is probably harder than you think.* Find the Fourier expansion of

$$g(x) = e^x$$

on $[-\pi,\pi]$.