Worksheet 3 (Jan. 27)

DIS 119/120 GSI Xiaohan Yan

1 Review

DEFINITIONS

- vector, zero vector, $\mathbb{R}^n$, addition, scalar multiplication;

- three things are identified: the numerical vector, the geometric vector, and the endpoint of the geometric vector;

- linear combination, span;

METHODS AND IDEAS

- Expanded **criterion for existence**: the constant column is a linear combination (i.e. in the **span**) of the variable coefficient columns $\Leftrightarrow$ solutions exist for a linear system $\Leftrightarrow$ the last column in REF is not pivotal.

Remark 1. One implication of the expanded criterion above is that, if the coefficient matrix has pivot in each **row** already, which means the column vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ span the entire $\mathbb{R}^m$, the associated linear system is always consistent, no matter what the constant vector $\mathbf{b}$ is. In particular, this is **impossible** when $n < m$.

2 Problems

Example 1. *We will do this together.* Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, where

$$\mathbf{b} = \begin{bmatrix} 9 \\ 1 \\ 10 \end{bmatrix}, \mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}.$$

After you have solved this, think: what does it mean geometrically? Are there easier solutions?

Example 2. True or false.

( ) The 3-dimensional zero vector $\mathbf{0}$ is a linear combination of the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ as in Example 1.

( ) The columns of an $m \times n$ matrix A span the entire $\mathbb{R}^m$ if and only if $m = n$.

( ) $\mathbf{b} \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ if and only if the augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$ has the last column as one of its pivot columns.

Example 3. Find the values of h such that $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{v} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}.$$

Example 4. *With a view toward future lectures.* Graph the following spans in $\mathbb{R}^2$:

$$(1)\,\text{span}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}; \qquad (2)\,\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\};$$

$$(3)\,\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}; \qquad (4)\,\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}.$$

How is the 4th case different, and why?