Worksheet 29 (April 23)

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1 Review

DEFINITIONS

- fundamental solution set, fundamental solution matrix.

METHODS AND IDEAS

Theorem 1. *Consider the 1st-order linear ODE system*

$$\mathbf{x}' = A\mathbf{x},$$

with A being a diagonalizable $n \times n$ matrix. Assume $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is an eigenbasis of A, and denote by $\{\lambda_1, \cdots, \lambda_n\}$ the corresponding eigenvalues. Then

$$\left\{e^{\lambda_1 t}\mathbf{v}_1, \cdots, e^{\lambda_n t}\mathbf{v}_n\right\}$$

is a fundamental solution set. In other words, the general solution is

$$\mathbf{x}(t) = c_1 \cdot e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n \cdot e^{\lambda_n t}\mathbf{v}_n.$$

Remark 1. When A is diagonalizable over $\mathbb{C}$, we have a similar theorem. More precisely, let's assume $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ are the two complex eigenvalues, and $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$ and $\mathbf{v}_2 = \mathbf{a} - i\mathbf{b}$ be the complex eigenvectors (complex eigenvalues and eiigenvectors come in pairs). Then,

$$\left\{e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b}, e^{\alpha t}\cos\beta t\mathbf{b} + e^{\alpha t}\sin\beta t\mathbf{a}, e^{\lambda_3 t}\mathbf{v}_3, \cdots, e^{\lambda_n t}\mathbf{v}_n\right\}$$

is a fundamental solution set. Note that for the first two terms, we are simply taking the real and imagnary parts of $e^{\lambda_1 t}\mathbf{v}_1$:

$$e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b} = \mathrm{Re}(e^{\lambda_1 t}\mathbf{v}_1) = \mathrm{Re}(e^{(\alpha+i\beta)t}(\mathbf{a} + i\mathbf{b})),$$

$$e^{\alpha t}\cos\beta t\mathbf{b} + e^{\alpha t}\sin\beta t\mathbf{a} = \mathrm{Im}(e^{\lambda_1 t}\mathbf{v}_1) = \mathrm{Im}(e^{(\alpha+i\beta)t}(\mathbf{a} + i\mathbf{b})).$$

2 Problems

Example 1. Consider the 1st order homogeneous linear system of ODE

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \mathbf{x}(t).$$

(a) Find a fundamental solution matrix of the system.

(b) Solve the initial value problem $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

(c) Sketch the trajectory of $\mathbf{x}(t)$ in (b).

Example 2. Find the general solution of the linear ODE

$$y'''(t) + y(t) = 0.$$