Worksheet 27 (April 16)

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1 Review

Theorem 1. *The solution set of the second-order linear homogeneous ODE*

$$a \cdot y''(t) + b \cdot y'(t) + c \cdot y(t) = 0$$

is a 2-dimensional vector space. More precisely, let λ_1 *and* λ_2 *be the two roots of the quadratic equation* $a\lambda^2 + b\lambda + c = 0$*, then the general solution is given by*

- ***if*** λ_1 ***and*** λ_2 ***are two distinct real numbers***$(b^2 - 4ac > 0)$*,*

$$y(t) = c_1 \cdot e^{\lambda_1 t} + c_2 \cdot e^{\lambda_2 t}, \qquad c_1, c_2 \in \mathbb{R};$$

- ***if*** λ_1 ***and*** λ_2 ***are the same real number***$(b^2 - 4ac = 0)$*,*

$$y(t) = c_1 \cdot e^{\lambda_1 t} + c_2 \cdot te^{\lambda_2 t}, \qquad c_1, c_2 \in \mathbb{R};$$

- ***if*** λ_1 ***and*** λ_2 ***are two complex numbers***$(b^2 - 4ac < 0)$*, denote* $\lambda_1 = u + iv$ *and* $\lambda_2 = u - iv$*,*

$$y(t) = c_1 \cdot e^{ut} \cos vt + c_2 \cdot e^{ut} \sin vt, \qquad c_1, c_2 \in \mathbb{R}.$$

Remark 1. Intuitively, the trigonometric functions come from the Euler formula $e^{i\theta} = \cos\theta + i \sin\theta$. Following the pattern of the first two cases, the two basis functions in the third case should have been $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. However, by the Euler formula,

$$e^{\lambda_1 t} = e^{ut+ivt} = e^{ut} \cdot e^{ivt} = e^{ut}(\cos vt + i \sin vt), \quad e^{\lambda_2 t} = \cdots = e^{ut}(\cos vt - i \sin vt),$$

so we might as well take the real and imaginary parts to be the basis functions.

2 Problems

Example 1. Solve the equations
(a) $2y''(t) - 5y'(t) + 3y(t) = 0$;
(b) $2y''(t) - 8y'(t) + 8y(t) = 0$;
(c) $y''(t) - 4y'(t) + 5y(t) = 0$.

Example 2. Consider the equation in (a) of the previous example. Denote by V its solution set. Prove that the linear transformation

$$\begin{aligned} T: \quad V &\longrightarrow \mathbb{R}^2 \\ y(t) &\longmapsto \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} \end{aligned}$$

is an isomorphism.

Hint: You already know a basis of V.

Example 3. Suppose that for $c \in \mathbb{R}$,

$$y''(t) + 2c \cdot y'(t) + c^2 \cdot y(t) = 0,$$

has a solution with $y(0) = 0, y(1) = 1, y(2) = 4$. Find the value of c or disprove existence.