Worksheet 26 (April 14)

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1 Review

DEFINITIONS

- Reduced SVD, full SVD;

- second-order, linear, homogeneous, constant-coefficient, ordinary, differential equations;

- linear algebra point of view of linear differential equations.

METHODS AND IDEAS

Theorem 1. *For any $m \times n$ matrix A with $\operatorname{rank} A = r$, $\exists$ diagonal matrix $\hat{\Sigma}_{r\times r}$, matrices with orthonormal columns $\hat{U}_{m\times r}$ and $\hat{V}_{n\times r}$ such that*

$$A = \hat{U}\hat{\Sigma}\hat{V}^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T. \qquad \textit{(Reduced SVD)}$$

Moreover, we may complete $\hat{U}$ and $\hat{V}$ into orthogonal matrices $U_{m\times m}$ and $V_{n\times n}$, and $\hat{\Sigma}$ into a "diagonal" but non-square matrix $\Sigma_{m\times n}$, such that

$$A = U\Sigma V^T. \qquad \textit{(Full SVD)}$$

When A is a symmetric matrix, the latter is the orthogonal diagonalization.

Theorem 2. *The solution set S of the 2nd-order homogeneous linear ODE with constant coefficients*

$$ay'' + by' + cy = 0,$$

where $a \neq 0 \in \mathbb{R}, b, c \in \mathbb{R}$, is a 2-dimensional subspace of the vector space of all smooth functions. Moreover, the initial value problems have unique solutions. In other words, for any initial value condition $y(0) = p, y'(0) = q$, there exists a unique function $y = y(x)$ satisfying both the ODE and the initial value conditions.

2 Problems

Example 1. *Example 3 of Worksheet 24.* Find both the reduced and the full SVD of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix}.$$

Example 2. Consider the second-order linear ODE

$$y'' + y = 0.$$

(1) Check that $f(x) = \sin x$ and $g(x) = \cos x$ are both solutions of the equation. (2) Solve the initial value problem $y(0) = 2, y'(0) = -1$. (3) Solve the initial value problem $y(\pi) = 2, y'(\pi) = -1$. (4) Find all solutions of the ODE.