Worksheet 24 (April 9)

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1 Review

DEFINITIONS

- symmetric matrices, orthogonally diagonalizable;

- singular value (of non-square matrices).

METHODS AND IDEAS

Theorem 1. (Properties of Symmetric Matrices)
Let A be a symmetric $n \times n$ matrix, then:
(a) all eigenvalues of A are real;
(b) eigenspaces of different eigenvalues of A are orthogonal;
(c) there is a basis of $\mathbb{R}^n$ consisting of orthonormal eigenvectors of A (which means A is orthogonally diagonalizable).

In fact the converse of (c) is also true: if a square matrix is orthogonally diagonalizable, it must be symmetric, too.

Method 1. (Algorithm of Singular Value Decomposition)
Assume that we are given an $m \times n$ matrix A with $m > n$.

1 *Find eigenvalues of the symmetric matrix A^TA, which are all real and non-negative. Denote them by $\lambda_1, \cdots, \lambda_n$. Assume that the first r of them are strictly positive and the rest are zero.*

2 *Find orthonormal eigenvectors of A^TA corresponding to the strictly positive eigenvalues $\lambda_1, \cdots, \lambda_r$. Denote them by $\mathbf{v}_1, \cdots, \mathbf{v}_r$.*

3 *Let $\sigma_i = \sqrt{\lambda_i}$ for $i = 1, \cdots, r$. These are the* ***singular values*** *of A.*

4 *Let $\mathbf{u}_i = \frac{1}{\sigma_i}A\mathbf{v}_i$ for $i = 1, \cdots, r$. Then,*

$$A = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \cdots + \sigma_r\mathbf{u}_r\mathbf{v}_r^T.$$

This is the ***singular value decomposition****(SVD) of A.*

Note that in this case rank $A = r$. Moreover, $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ is an orthonormal basis of Row(A), while $\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$ is an orthonormal basis of Col(A).

2 Problems

Example 1. Diagonalize the matrix below with orthogonal matrices.

$$S = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Example 2. Let us still consider the matrix S from the previous example. Is there a vector $\mathbf{x} \in \mathbb{R}^3$ such that $\mathbf{x}^T S \mathbf{x} < 0$?

Example 3. Find SVD of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix}.$$