Worksheet 23 (April 2)

DIS 119/120 GSI Xiaohan Yan

1 Review

METHODS AND IDEAS

Theorem 1. *The row rank is equal to the column rank for any matrix. In fact we have*

$$\dim \mathrm{Row}(A) + \dim \mathrm{Nul}(A) = n \text{ and } \dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) = n,$$

where the former comes from the orthogonality, and the latter comes from the Rank-Nullity theorem.

Remark 1.
$$\mathrm{Row}(A) = \mathrm{Nul}(A)^{\perp}, \mathrm{Col}(A) = \mathrm{Nul}(A^T)^{\perp}.$$

Remark 2. Another way to see the theorem is by row reduction. In fact, both the row rank and the column rank are preserved by row reductions. (Why?) So we may reduce the theorem to the case of RREF. But in RREF both ranks are equal to the number of pivots.

Theorem 2. *Otrhgonal matrices preserve the inner product. In other words, given an orthogonal matrix U, we have*

$$U\mathbf{x} \cdot U\mathbf{y} = \mathbf{x} \cdot \mathbf{y}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Remark 3. In particular, this gives $||U\mathbf{x}|| = ||\mathbf{x}||$ if we take $\mathbf{x} = \mathbf{y}$.

2 Problems

Example 1. True or false.

( ) Let U be an orthogonal matrix, then $\det(U) = 1$.

( ) Let U be an orthogonal matrix and $\mathbf{x}$ a vector such that $U\mathbf{x}$ and $\mathbf{x}$ are linearly dependent, then $U\mathbf{x} = \pm\mathbf{x}$.

( ) If U is diagonal and orthogonal, then U must be an identity matrix.

( ) The Gram-Schmidt process produces from a linearly independent set $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ an orthogonal set $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ with the property that for each k,

$$\text{span}\{\mathbf{w}_1, \cdots, \mathbf{w}_i\} = \text{span}\{\mathbf{u}_1, \cdots, \mathbf{u}_i\}, \forall i = 1, 2, \cdots, k.$$

( ) For any two matrices A and B such that AB is well-defined,

$$\operatorname{rank} AB \leqslant \max\{\operatorname{rank} A, \operatorname{rank} B\}.$$

Example 2. Find an example or disprove existence:
a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $T^2 = T$ but is not an orthogonal projection.

Example 3. Find the best fitting model in the form $y = ax^2 + bx + c$ of the data points

$$(-1, 1), (0, 0), (1, 1), (2, 1).$$

Example 4. Find the values of a, b, c, d, e, f, g such that U is an orthogonal matrix:

$$\begin{pmatrix} 1 & c & e \\ a & \frac{\sqrt{2}}{2} & f \\ b & d & g \end{pmatrix}.$$