Worksheet 22 (March 31)

DIS 119/120   GSI Xiaohan Yan

1 Review

DEFINITIONS

- orthogonal matrix;

- least square solution of a linear system, existence and uniqueness?

METHODS AND IDEAS

Theorem 1. (Orthogonal Projection)
Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ *be an* ***orthonormal basis*** *of the subspace* $W \subset \mathbb{R}^n$. *Then the orthogonal projection* $\mathrm{Proj}_W \mathbf{v}$ *of a vector* $\mathbf{v} \in \mathbb{R}^n$ *is*

$$\mathrm{Proj}_W \mathbf{v} = UU^T \mathbf{v},$$

where U *is the matrix whose columns are* $\mathbf{u}_1, \cdots, \mathbf{u}_n$. *In other words, the standard matrix (i.e. the matrix under the standard basis) of the linear transformation* Proj_W *is* UU^T.

Remark 1. The kernel and image of Proj_W are $W^\perp$ and W respectively.

Remark 2. Proj_W can only have two eigenvalues 0 and 1. Moreover, $E_1 = W$ and $E_0 = W^\perp$

Method 1. (Gram-Schmidt Process)
Given a (not necessarily orthogonal) basis $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ *of* $W \subset \mathbb{R}^n$, *we construct inductively an orthogonal basis* $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ *of* W *out of* $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$, *by taking*

$$\mathbf{w}_1 = \mathbf{u}_1, \mathbf{w}_{i+1} = \mathbf{u}_{i+1} - \left(\frac{\mathbf{u}_{i+1} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 + \cdots + \frac{\mathbf{u}_{i+1} \cdot \mathbf{w}_i}{\mathbf{w}_i \cdot \mathbf{w}_i} \mathbf{w}_i \right).$$

In other words, we remove from u_{i+1} *its "orthogonal projection to the previous vectors" to obtain* w_{i+1}.

Remark 3. If you want an orthonormal basis, divide the orthogonal basis vectors by their norms.

2 Problems

Example 1. True or false.

( ) The dot product $\mathbf{x} \cdot \mathbf{y}$ is the only entry of the 1×1 matrix $\mathbf{x}^T\mathbf{y}$.

( ) The standard matrix of an orthogonal projection is always diagonalizable.

( ) If the linear system $A\mathbf{x} = \mathbf{b}$ is consistent, the set of its least square solutions is exactly its solution set.

Example 2. Consider

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 1 \\ 4 & -1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Find the least square solutions of $A\mathbf{x} = \mathbf{b}$.
(b) Find the best approximation (orthogonal projection) of $\mathbf{b}$ to $\mathrm{Col}(A)$.

Example 3. Find an orthonormal basis of the subspace $W = \{(x, y, z, w) | x + y + z + w = 0\}$ of $\mathbb{R}^4$.