Worksheet 20 (March 19)

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1 Review

DEFINITIONS

- metric geometry of $\mathbb{R}^n$: inner product, length, angle;

- orthogonal set, orthonormal set, orthogonal basis;

- orthogonal complement, orthogonal projection.

METHODS AND IDEAS

Theorem 1. *If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are orthogonal vectors, then they are linearly independent.*

Theorem 2. (Orthogonal Projection)
*Let $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$ be an **orthogonal** basis of subspace $W \subset \mathbb{R}^n$, then for any vector $\mathbf{y} \in \mathbb{R}^n$, its orthogonal projection to W is*

$$\hat{\mathbf{y}} = \operatorname{Proj}_W \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{w}_k}{\mathbf{w}_k \cdot \mathbf{w}_k}\mathbf{w}_k.$$

Remark 1. The orthogonal projection $\hat{\mathbf{y}}$ is the closest to $\mathbf{y}$ among all vectors in W. Moreover, it is the unique vector in W such that $\mathbf{y} - \hat{\mathbf{y}}$ is orthongonal to W. In other words, the decomposition of any vector $\mathbf{y}$ into the sum of W and $W^\perp$ is unique, and it is exactly $\hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}})$.

Remark 2. When W in the theorem is the full subspace $\mathbb{R}^n$, $\mathcal{W} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\}$ is a basis of $\mathbb{R}^n$ and thus the formula gives an easy way to compute the $\mathcal{W}$-coordinate of $\mathbf{y}$, i.e.

$$[\mathbf{y}]_{\mathcal{W}} = \begin{pmatrix} \frac{\mathbf{y}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1} \\ \vdots \\ \frac{\mathbf{y}\cdot\mathbf{w}_k}{\mathbf{w}_1\cdot\mathbf{w}_k} \end{pmatrix}.$$

2 Problems

Example 1. True or false.

( ) If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are orthogonal, $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\|$.

( ) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a set of orthogonal vectors if and only if any two vectors of it are orthogonal to each other.

( ) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a set of linearly independent vectors if and only if any two vectors of it are linearly independent to each other.

( ) Let $W \subset \mathbb{R}^n$ be a subspace, then its orthogonal complement $W^\perp$ is a subspace of $\mathbb{R}^n$ of complementary dimension.

( ) Let $W \subset \mathbb{R}^n$ be a subspace, then $(W^\perp)^\perp = W$.

( ) Let $W \subset \mathbb{R}^n$ be a subspace and $W^\perp$ be its orthogonal complement. If $\mathbf{v}$ is in both W and $W^\perp$, then $\mathbf{v}$ must be the zero vector.

Example 2. Consider the vector

$$\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \in \mathbb{R}^2.$$

(a) Compute $\mathbf{v} \cdot \mathbf{e}_1$ and $\mathbf{v} \cdot \mathbf{e}_2$.
(b) Suppose $\mathbf{u} \in \mathbb{R}^2$ is a unit vector satisfying $\mathbf{u} \cdot \mathbf{v} = 2$. Find $\mathbf{u}$.
(c) Find the area of the triangle formed by the origin and the endpoints of $\mathbf{u}$ and $\mathbf{v}$.

Example 3. Let W be the plane in $\mathbb{R}^3$ given by $x + y + z = 0$.
(a) Find the orthogonal projection of $\mathbf{x} = (7, -1, 3)^T$ to W.
(b) Find all $\mathbf{y}$ whose orthogonal projection to W is $(2, 2, -4)^T$.