Worksheet 17 (March 12)

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1 Review

DEFINITIONS

- similar matrices, eigenvalues and eigenvectors of similar matrices;

- diagonalization.

METHODS AND IDEAS

Theorem 1. (Diagonalizability)
An $n \times n$ matrix A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors (which thus form a basis of $\mathbb{R}^n$) $\Leftrightarrow$ the sum of geometric multiplicities of eigenvalues of A is n $\Leftrightarrow$ for all eigenvalues of A, the algebraic multiplicity is equal to the geometric multiplicity.

Remark 1. (Algorithm for diagonalization)
(1) Given matrix A, we first find its characteristic polynomial and solve for eigenvalues.
(2) For each eigenvalue λ of A, we find a basis of the eigenspace $E_\lambda = \mathrm{Nul}(A - \lambda I)$.
(3) If $\dim E_\lambda$ is equal to the algebraic multiplicity of λ for all λ, A is diagonalizable. Take P as the matrix whose columns are the basis vectors we found in those E_λ in the previous step, then $D = P^{-1}AP$ gives a diagonalization of A, and D is the diagonal matrix whose diagonal entries are eigenvalues of the columns of P.

2 Problems

Example 1. Find an example or disprove existence:
(a) Diagonalizable 3×3 matrix M that is not invertible.

(b) Diagonalizable 3×3 matrix M with 2 distinct eigenvalues.

Example 2. Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix},$$

i.e. find invertible matrix P and diagonal matrix D such that $D = P^{-1}AP$.

Example 3. *What is described in this example is entirely hypothetical.* YiFang and FengCha are two boba shops in Berkeley. Denote by $Y(t)$ and $F(t)$ the numbers of customers of these two shops on day t. An economist who newly learned some linear algebra formulated the following recursive relation between $Y(t)$ and $F(t)$:

$$\begin{pmatrix} Y(n+1) \\ F(n+1) \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} Y(n) \\ F(n) \end{pmatrix}$$

Assume that this model is correct, and that $Y(0) = 34$ and $F(0) = 32$, i.e. on day 0 they have 34 customers and 32 customers respectively. Can you help the economist to compute $Y(5)$ and $F(5)$?