Worksheet 16 (March 10)

DIS 119/120 GSI Xiaohan Yan

1 Review

DEFINITIONS

- eigenvector, eigenvalue;
- characteristic polynomial;
- eigenspace, algebraic multiplicity, geometric multiplicity.

METHODS AND IDEAS

Theorem 1. (Fundamental Theorem of Algebra)
A polynomial of degree n has exactly n complex roots, and thus at most n real roots.

Remark 1. An $n \times n$ matrix has at most n **real eigenvalues** (and exactly n **complex eigenvalues**), counted with algebraic multiplicity.

Theorem 2. *Eigenvectors of different eigenvalues are linearly independent.*

Remark 2. As a corollary, one can find n linearly independent eigenvectors for an $n \times n$ matrix A in the following two cases:

- A has n distinct eigenvalues;
- A has less than n eigenvalues $\lambda_1, \dots, \lambda_k (k < n)$, but all of the eigenvalues have full geometric multiplicities, i.e.

$$\dim E_{\lambda_1} + \dim E_{\lambda_2} + \cdots + \dim E_{\lambda_k} = n.$$

Note that in general, the **geometric multiplicity** is smaller than or equal to the **algebraic multiplicity**, so the sum above might be strictly smaller than n.

2 Problems

Example 1. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

(a) Find all eigenvalues of A, and determine their algebraic multiplicities.
(b) For each eigenvalue, find a basis for its eigenspace, and determine its geometric multiplicity.
(c) Note that A is 3×3. Are there 3 linear independent eigenvectors of A?

Example 2. Find the values of c and d such that $(1, 1, 1)^T$ is an eigenvector of of the matrix

$$A = \begin{pmatrix} 2 & 1 & c \\ -2 & 4 & -2 \\ 0 & b & 7 \end{pmatrix}.$$

Example 3. Find all real eigenvalues and an eigenvector for each real eigenvalue of the following linear transformations of $\mathbb{R}^2$.
(a) Scale the x-direction by 2. (b) Rotation counterclockwise by $\pi/2$. (c) Reflection across $y = x$. (b) Sheer transformation sending $\mathbf{e}_1$ to $\mathbf{e}_1$ but $\mathbf{e}_2$ to $\mathbf{e}_1 + \mathbf{e}_2$.

Example 4. Let M be a 2×2 matrix with two distinct eigenvalues 2 and 4. Find $\det(M)$.