Worksheet 14 (March 5)

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1 Review

METHODS AND IDEAS

Theorem 1. (rank-nullity)
Let $T : V \longrightarrow W$ be a linear transformation, we will always have

$$\dim \ker(T) + \dim \mathrm{Im}(T) = \dim V.$$

Theorem 2. (coordinates under base change)
Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of vector space V, then the coordinates of the same vector $\mathbf{x} \in V$ under $\mathcal{B}$ and $\mathcal{C}$ are related by

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}},$$

where the base change matrix is given by

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} [\mathbf{b}_1]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{pmatrix}.$$

Theorem 3. (linear transformation under base change)
Let $\mathcal{A}$ and $\mathcal{B}$ be two bases of vector space V, $\mathcal{C}$ and $\mathcal{D}$ be two bases of vector space W, and $T : V \longrightarrow W$ a linear transformation. Then the matrices of the same linear transformation T under different bases are related by

$$_{\mathcal{D}}[T]_{\mathcal{B}} = P_{\mathcal{D}\leftarrow\mathcal{C}}\ {}_{\mathcal{C}}[T]_{\mathcal{A}} \cdot P_{\mathcal{A}\leftarrow\mathcal{B}}.$$

In particular, if $V = W$ and $\mathcal{A}$ and $\mathcal{B}$ are two bases of V,

$$_{\mathcal{B}}[T]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{A}}\ {}_{\mathcal{A}}[T]_{\mathcal{A}} \cdot P_{\mathcal{A}\leftarrow\mathcal{B}}.$$

2 Problems

Example 1. True or false.

( ) Base change matrices $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are always invertible.

( ) If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three bases of the vector space V, then

$$P_{\mathcal{C}\leftarrow\mathcal{A}} = P_{\mathcal{C}\leftarrow\mathcal{B}} \cdot P_{\mathcal{B}\leftarrow\mathcal{A}}.$$

( ) If M and N are two matrices of a linear transformation $T : V \longrightarrow W$ (relative to different bases), then $\operatorname{rank} M = \operatorname{rank} N$.

Example 2. *Example 3 from the previous worksheet.* Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1.\mathbf{b}_2, \mathbf{b}_3\}$ be two different bases of $\mathbb{R}^3$, where

$$\mathbf{a}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(a) Find $[\mathbf{a}_i]_{\mathcal{B}}$ for $i = 1, 2, 3$.
(b) If $[\mathbf{x}]_{\mathcal{A}} = (1, 1, 1)^T$, find $[\mathbf{x}]_{\mathcal{B}}$.
(c) If $[\mathbf{y}]_{\mathcal{A}} = [\mathbf{y}]_{\mathcal{B}}$, find $\mathbf{y}$.

Example 3. *Example 4 from the previous worksheet.* Consider the linear transformation $T : \mathbb{P}_2 \to M_{2\times 2}(\mathbb{R})$ defined as

$$T(a + bx + cx^2) = \begin{pmatrix}3a + b & b + c\\ -2b & a + b + c\end{pmatrix}.$$

(a) Consider the basis $\mathcal{B} = \{1+x, x, x^2-x\}$ of $\mathbb{P}_2$ and the basis $\mathcal{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ of $M_{2\times 2}(\mathbb{R})$. Compute ${}_{\mathcal{B}}[T]_{\mathcal{E}}$.
(b) Consider instead the basis $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ of $M_{2\times 2}(\mathbb{R})$, where

$$C_1 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, C_2 = \begin{pmatrix}4 & 1\\ -2 & 2\end{pmatrix}, C_3 = \begin{pmatrix}1 & 1\\ -2 & 1\end{pmatrix}, C_4 = \begin{pmatrix}-1 & 0\\ 2 & 0\end{pmatrix}.$$

Compute ${}_{\mathcal{B}}[T]_{\mathcal{C}}$.
(c) Is T one-to-one? Onto?