Worksheet 13 (March 3)

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1 Review

DEFINITIONS

- coordinate mapping;

- matrix of linear transformation relative to bases on domain and codomain;

METHODS AND IDEAS

Theorem 1. *Any two vector spaces of the same dimension are isomorphic, since any vector space of dimension n is isomorphic to $\mathbb{R}^n$ under the coordinate mapping under a basis.*

Note that vector spaces of different dimensions can never be isomorphic. This is because isomorphism preserves linear independence and spanning property, and thus always sends a basis to a basis. But bases of vector spaces of different dimensions contain different number of vectors.

2 Problems

Example 1. True or false. In the last three statements, S denotes the vector space of all smooth (infinitely differentiable) functions over $[0, 1]$. In other words, you do not need to worry about differentiability of elements of S

( ) There exists a basis $\mathcal{B}$ of $\mathbb{P}_2$ such that $1 + x$ has coordinate $(1, 1, 1)^T$ while $x + x^2$ has coordinates $(-3, -3, -3)$.

( ) Let V be a 5-dimensional vector space and $\mathbf{v}_1, \mathbf{v}_2$ be two linearly independent vectors in V, then there exists three other vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ in V such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis of V.

( ) Let W be a 2-dimensional vector space and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ be five vectors in W, then we can take two of these vectors to form a basis of W.

( ) Let $\mathcal{A}, \mathcal{B}$ be two bases of the vector space V, then for any vector $\mathbf{v} \in V$,

$$[\mathbf{v}]_{\mathcal{A}} = [\mathcal{B}]_{\mathcal{A}} \cdot [\mathbf{v}]_{\mathcal{B}},$$

where $[\mathcal{B}]_{\mathcal{A}}$ is the square matrix whose columns are $\mathcal{A}$-coordinate vectors of the vectors of $\mathcal{B}$.

( ) There is no isomorphism $T : \mathbb{P}_2 \to \mathbb{R}^2$.

( ) Let $T : S \to S$ be the linear transformation $T(f(x)) = f'(x)$, then T is surjective.

( ) Let $T : S \to S$ be the linear transformation $T(f(x)) = \int_0^x f(s)ds$, then T is surjective.

( ) Let $T : S \to S$ be the linear transformation $T(f(x)) = f''(x)$, then T has a two-dimensional kernel.

Example 2. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1.\mathbf{b}_2\}$ be two different bases of $\mathbb{R}^2$ where

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{b}_1 = \begin{pmatrix}2\\5\end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix}1\\3\end{pmatrix}.$$

(a) Let $\mathbf{x} = \begin{pmatrix}-1\\-1\end{pmatrix}$. Write $\mathbf{x}$ as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$.

(b) Compute $[\mathbf{x}]_{\mathcal{E}}$ and $[\mathbf{x}]_{\mathcal{B}}$.

(c) Let $B = \begin{pmatrix}2 & 1\\5 & 3\end{pmatrix}$. Check that $[\mathbf{x}]_{\mathcal{E}} = A[\mathbf{x}]_{\mathcal{B}}$. Can you explain the reason behind this?

(d) Now let's generalize this result. Let $\mathcal{A} = \{\mathbf{a}_1.\mathbf{a}_2\}$ be yet another basis of $\mathbb{R}^2$. Given that

$$[\mathbf{b}_1]_{\mathcal{A}} = \begin{pmatrix}23\\45\end{pmatrix}, \qquad [\mathbf{b}_2]_{\mathcal{A}} = \begin{pmatrix}89\\67\end{pmatrix},$$

find $[\mathbf{x}]_{\mathcal{A}}$.

Example 3. *One more question about base change.* Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1.\mathbf{b}_2, \mathbf{b}_3\}$ be two different bases of $\mathbb{R}^3$, where

$$\mathbf{a}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix} \mathbf{b}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(a) Find $[\mathbf{a}_i]_{\mathcal{B}}$ for $i = 1, 2, 3$.
(b) If $[\mathbf{x}]_{\mathcal{A}} = (1, 1, 1)^T$, find $[\mathbf{x}]_{\mathcal{B}}$.
(c) If $[\mathbf{y}]_{\mathcal{A}} = [\mathbf{y}]_{\mathcal{B}}$, find $\mathbf{y}$.

Example 4. Consider the linear transformation $T : \mathbb{P}_2 \to M_{2\times 2}(\mathbb{R})$ defined as

$$T(a + bx + cx^2) = \begin{pmatrix}3a + b & b + c\\ -2b & a + b + c\end{pmatrix}.$$

(a) Consider the basis $\mathcal{B} = \{1+x, x, x^2-x\}$ of $\mathbb{P}_2$ and the basis $\mathcal{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ of $M_{2\times 2}(\mathbb{R})$. Compute ${}_{\mathcal{B}}[T]_{\mathcal{E}}$.
(b) Consider instead the basis $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ of $M_{2\times 2}(\mathbb{R})$, where

$$C_1 = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, C_2 = \begin{pmatrix}4 & 1\\-2 & 2\end{pmatrix}, C_3 = \begin{pmatrix}1 & 1\\-2 & 1\end{pmatrix}, C_4 = \begin{pmatrix}-1 & 0\\2 & 0\end{pmatrix}.$$

Compute ${}_{\mathcal{B}}[T]_{\mathcal{C}}$.
(c) Is T one-to-one? Onto?